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CHARLES DE BOUVELLES' INTRODUCTION TO STAR POLYGONS

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Abstract

Charles de Bouvelles (1471–1553) was a canon of the cathedral of Noyon, where he dedicated most of his energies to scholarly pursuits centering around mathematics, philosophy and theology. His introduction to geometry (1511), written for craftsmen and artisans, was the first geometry text to be published in French instead of Latin. The work was not very successful and, thus, was re-elaborated and republished in 1542. The present text investigates Bouvelles' presentation of star polygons in the 1542 text, relating it to his investigations of figurate numbers and placing it in its cultural context.

Keywords: History of Geometry, Star Polygons, Charles de Bouvelles.

[A INTRODUÇÃO A POLÍGONOS ESTRELADOS DE CHARLES DE BOUVELLES]

Resumo

Charles de Bouvelles (1471–1553) foi cônego do catedral de Noyon, onde ele se dedicou a maior parte das suas energias a atividades eruditas centradas em matemática, filosofia e teologia. Sua introdução à geometria de 1511, escrita para artisões, foi o primeiro texto de geometria a ser publicado em francês em vez de latim. Teve pouco sucesso e, assim, foi reescrito e republicado em 1542. O presente text investiga a apresentação de Bouvelles de polígonos estrelados no texto de 1542, comparando-a com as suas investigações sobre números figurados e colocando-a no seu contexto cultural.

Palavras-chave: História da Geometria, Polígonos estrelados, Charles de Bouvelles.

Introduction

Figure 1. Bouvelles' stained-glass window. **Source:** Mimesi (2012).

Renowned in his lifetime as one of France's foremost thinkers, Charles de Bouvelles (1471–1553)[1] has been, for the most part, gobbled up by the vicissitudes of history. Still, we have come across him once before (Fossa, 2020) as the author of a very nice proof by exemplification [see also Fossa (2021)] of the proposition that every perfect number is triangular, which is contained in his little work *Liber de perfectis numeris*. [Bouillus (1510)]. In any case, he was born into an aristocratic family in Soyécourt, a small community about 24 kilometers east of present-day Amiens and about 120 kilometers north of Paris. The principal city of the region, Amiens was known at one time by the Romans as Samarobriva and, thus, Bouvelles was known in Latin as Carolus Bouillus Samarobrinus. He went to university at Paris, but apparently did not finish before the city was sorely hit by the plague, which induced him to travel about the country, as well as to visit other European centers. Upon returning to Paris, he was ordained a priest and taught at the *Collège du Cardinal-Lemoine*, a *collège* associated with the University of Paris, where, in fact, he had been a student. Later he was appointed a canon of the cathedral of Noyon, a town located about 60 kilometers to the southeast of Amiens. As a canon, his principal duties would have been teaching, giving assistance to the poor and, especially, organizing the choir. He was also elector of the *Collégiale Saint-Quentin* (now *Basilique Saint-Quentin*) in the town of Saint-Quentin, about 80 kilometers to the east of Amiens, named after the 3rd century martyr Caius Quintinus (?–287), also known as Quentin d'Amiens. In 1521, Bouvelles dedicated a stained-glass window to the *Basilique*, depicting Saint Catherine (see Figure 1). Bouvelles himself is pictured kneeling in the bottom left-hand corner (see Figure 2). Apparently, his ecclesiastical duties at Noyon were very

Figure 2. Detail of Figure 1.

[1] His dates are also given as (1475-1566). I follow O'Connor e Robertson (2002).

light, due to the support given to him by the local bishop, who wanted to encourage him in his scholarly pursuits. These pursuits centered around mathematics, philosophy and theology. One of his abiding interests in geometry was that of squaring the circle – in his time still an open question –, which earned him a brief mention in Augustus De Morgan's *A Budget of Paradoxes* (see De Morgan, 1915)).

An Arithmetical Propaedeutic

All but confessing himself to be a Pythagorean – albeit a Christian Pythagorean –, Bouvelles initiates his *Liure ſingulier & vtile, tovchant l'art et practique de Geometrie* (1542)[2] – henceforth *Livre singulier* – by declaring geometry to be subservient to and dependent on arithmetic. The latter, he explains – in allusion to the Pythagorean *tetraktys* – consists of four principles, the numbers 1, 2, 3 and 4, whose sum is the perfect[3] number 10 which comprises all things. These noetic principles are instantiated corporally by the corresponding four principles of geometry, to wit: the point, the line, the surface and the solid body.

Figurate numbers, that is collections of units displayed as geometrical figures, had various uses in Antiquity. They may have also suggested certain analogies to Bouvelles. *Ordinary figurate numbers* are built up from the unit by subjoining to each member of the sequence a *gnomon*, which transforms each number to the next one in the sequence. For regular polygons, the gnomon has the shape of the polygon minus two sides. Figure 3 depicts the first few triangular numbers with their gomons indicated.

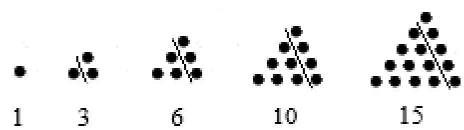

Figure 3. First five triangular numbers.

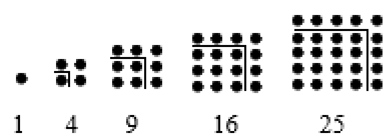

Figure 4. First five square numbers.

[2] This is a rewritten version of [Bouillus] (1511), reputed to be the first treatise on geometry published in French. The reworked version treats pretty much of the same material with, however, some additions, and its layout is vastly improved, making it much more reader-friendly. On this, see also Oosterhoff (2017).

[3] The number 10 is perfect because of its completeness. This should not be confused with another kind of perfect numbers, those that are the sum of their aliquot parts.

Figure 4 does the same thing for square numbers.

One of the important features of figurate numbers is that there are multiple relations subsisting among them. Figure 5 shows two different decompositions of the fifth square number.

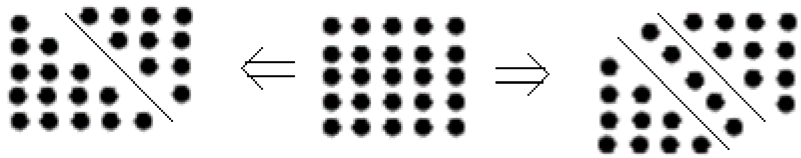

Figure 5. Two decompositions of the fifth square number.

The relation is obviously generalizable: a square number is equal to the corresponding triangular number and its predecessor; it is also equal to the corresponding linear number and two of the preceeding triangular number. By using indexed variables, which, of course, were not available to Bouvelles, this can be stated more precisely. To this end, let t_n be the n^{th} triangular number, s_n the n^{th} square number and l_n the n^{th} linear number. Then we have

$$s_n = t_n + t_{n-1} = l_n + 2t_{n-1}.$$

Now, all these types of figurate numbers have algebaic formulations[4], but these do not concern us at the moment. Rather, let's compare linear numbers with geometric lines. From the times of Euclid, at least, the geometric line has been considered as having no thickness. Linear numbers, in contrast, may be considered as "broad lines", much in the manner of Høyrup's (2002) interpretation of Babylonian calculation. Geometry is an idealization of real-world corporal relationships, whereas figurate numbers comprise a phenomological model for perceiving the more esoteric noetic relationships of arithmetic. Nonetheless, even though body is an imperfect instantiation of arithmetic and, thus, arithmetical results do not always carry over into geometry, arithmetical methods may suggest ways of investigating geometry. The case in point is that decomposition is an interesting way of investigating geometrical relationships.

It is, of course, quite easy to form *figurate star polygons* from the figurate polygons by appending to each side of the latter the appropriate triangular number. Herein the proceedure will be illustrated with regard to another kind of figurate number, *centered figurate numbers*. In contrast to ordinary figurate numbers, which grow from a common vertex (the unit) by adding a gnomon of two less sides than that of the polygon in question, centered figurate numbers grow by appending ever greater sizes of the polygon in question about a common center (the unit). Both proceedures depend on the fact that the unit is the

[4] $l_n = n$, $t_n = \frac{n(n+1)}{2}$ and $s_n = n^2$.

first member of every sequence of figurate polygons. We would probably consider this to be the result of expanding the definition of "polygon" in order to obtain a zero-sided "degenerate" polygon, but the Pythagoreans saw the unit as the arithmetical correlate of the all-encompassing One – the all-encompassing Godhead for Christian Pythagoreans – and, thus, it is no surprise that the unit is a triangle, a square, *etc*. and that, in each case, it gives rise to the other members of the sequence.

Now, Bouvelles does not treat of figurate numbers in *Livre singulier*, but he does investigate, under the name *arithmetice rose*[5], centered figurate numbers in his *Libellus de mathematicis rofis* (Bouillus, 1510a). Letting h_n be the n^{th} ordinary hexagonal number (1, 6, 15, 28, 45, 66, 91, …) and H_n the n^{th} centered hexagonal number (1, 7, 19, 37, 61, 91, 127, …), Bouvelles includes a depiction of H_3 in the margin of his text (Figure 6) and observes (Figure 7) that

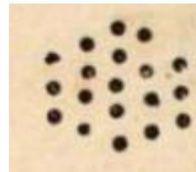

Figure 6. H_3.
Source: Bouillus (1510a), p. 180.

$$H_n - h_n = s_{n-1}.$$

Rofarum numeri	1	7	19	37	61	91	127
Arithmetici exagoni	1	6	15	28	45	66	91
Quadrate differentie	0	1	4	9	16	25	36

Figure 7. $H_n - h_n$.
Source: Bouillus (1510a), p. 182.

Interestingly, since the successive gnomons for, say, centered hexagonal numbers are naught but the successive multiples of six, there is a beautiful result for the n^{th} hexagonal number in terms of (ordinary) triangular numbers. That is, since $H_n = 1 + 6 + 12 + 18 + \cdots + (n-1)6 = 1 + 6(1 + 2 + 3 + \cdots + (n-1))$, we obtain[6]

$$H_n = 1 + 6t_{n-1}.$$

In order to obtain the centered hexagonal star number SH_n, we append a copy of the (ordinary)[7] triangular number t_{n-1} to each of the six sides of SH_n. Figure 8 (left side) shows SH_3 The body of the star is the centered hexagonal number $H_3 = 19$, whereas each arm thereof is $t_2 = 3$, making a total of 6×3 = 18 in the six arms. Figure 8 shows how this relation is maintained for all hexagonal numbers. The gnomon of the hexagonal body is shown in grey (18 grey stones) and the gnomon of each of the triangular arms is shown in yellow (18

[5] *Arithmetice* is a rare form for *arithmetica* and *rose* a rare form for *rosa*.
[6] For those who would like to maintain the triangular theme, t_1 may be substituted for the unit. The relation is clearly generalizable: let ${}^{m}P_n$ be the n^{th} m-gonal number, then ${}^{m}P_n = 1 + mt_{n-1}$. For those of the polytheistic persuasion, observe that ${}^{m}P_n \equiv 1 \pmod{m}$.
[7] Of course, centered triangular numbers may be used, but, for the first approach to star-making, it is more intuitively pleasing to use ordinary triangles.

yellow stones). Therefore, the difference does not change. Algebraically, this becomes obvious: Hexagonal body – 6(triangular arm) = $1 + 6t_{n-1} - 6(t_{n-1}) = 1$.

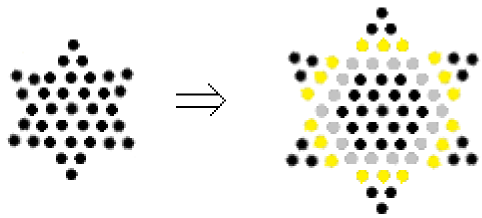

Figure 8. How to generate SH_4 from SH_3.

Right at the end of *Libellus de mathematicis rofis*, Bouvelles mentions another kind of centered polygonal number, resulting from removing the central stone (which corresponds to removing the initial unit from the polygonal sequence). The complete centered polygonal numbers he calls *arithmetice rofe centrum poffidentes*, those without the center stone, *arithmetice poligonie centro carentes*. Naturally, one can also take the central stone from the star figurate numbers, obtaining thereby an uncanny visual image (see Figure 9). The missing stone at the center of the figure reminds us of an eye and, thus, of the desembodied eye symbolism that has appeared (semi-independently) throughout history in many different cultures, from the Egyptian Eye of Horus to the eye in the triangle on the US dollar. It is, nevertheless, much more abstractly mystical, because, instead of presenting a likeness to the physical eye, it represents the all-seeing eye that is not itself seen.

The removal of the central stone, however, is not conceptually arbitrary, given the Christo-Pythagorean context in which it occurs. The central stone is, of course, the first member of the sequence of star figurate numbers of which it is a part. As such, it is a mathematical correlate of the Pythagorean One and the Christian Godhead. Philosophically, it represents Being, the underlying sustenance of everything that exists, but which is never found of itself, only as beings.

In any case, computationally, the star polygons with the center stone removed are easily calculated. Let, for example, SHE_n be the n^{th} figurate star hexagon with an eye. Then, clearly, SHE_n = (body of the (full) star minus the unit)+(arms of the star). But, as we have already seen, the arms are a unit less than the (full) body. Therefore, applying the formula for H_n to find the body of the star, we obtain

$$SHE_n = 12t_{n-1}.$$

Thus, as in this special case, in a centered polygon with the center stone removed, the arms, altogether, contain the same amount of stones as does the polygonal body.

Given that $t_{14} = 105$, we find that $SHE_{15} = 1260$, which is traditionally understood as the number of days in the Biblical (Daniel; Revelation) *time* (a year of 360 days), *times* (two years) *and half a time* (half a year). Thus, Figure 9 is a (Pythagorean) mathematical symbol of Christian doctrine.[8]

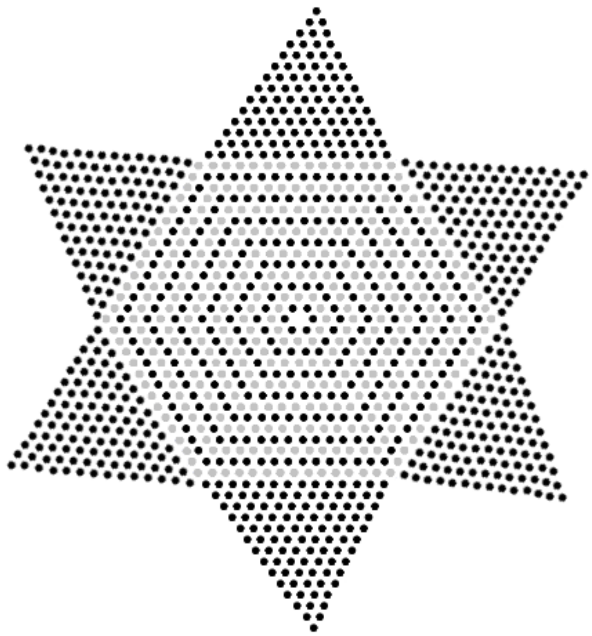

Figure 9. SHE_{15}.

The *Livre singulier*

I have already indicated, *en passant* (footnote 2), that the *Livre singulier* is a rewritten version of the earlier and rather unsuccessful *Geometrie en francoys*. Saito (2017) characterizes the work[9] as a practical, as opposed to a scientific, treatise. It is, in fact, not only composed in the vernacular, but also abandons the rather strict Euclidean proof format as it explains how to use the tools of geometry in order to draw geometrical diagrams and, further, is explicitly addressed to *auturgis, manúve operarijs* ("handymen, or manual workers"; probably better,

[8] There should, however, be no grey stones in the figure. I have, nonetheless, used two colors in the figure to highlight its mathematical structure.
[9] Saito (2017) uses the 1551 edition of the work.

"craftsmen") such as (see Saito (2017) and/or Oosterhoff (2017)) artisans working with problems in surveying, mechanics, pneumatics and hydrostatics. He further states that

> "*Confeci igitur Gallica lingua Geometricum Iſagogicum*
> *(I have therefore composed [the present] Introduction to Geometry in the French language).*"

In this regard, the *Livre singulier* seems to be both an introduction to geometrical constructions for those who wish to apprentice themselves to a master and a kind of handbook for the craftsman, especially regarding questions relating to proportionalities.

With all this in mind, we may now ask: Why did Bouvelles think it appropriate to include star ploygons in this introductory work? Aldo Brigaglia, Nicla Palladino & Maria Alessandra Vaccaro (2018) suggest that the main reason is because of their use in decorative motifs. That is, of course, consonant with the practical nature of the *Livre singulier* since pratice, or rather art (τέχνη), is traditionally understood as an activity that produces something – usually something physical. Even products like decorative motifs, however, have theoretical underpinnings. Some of Bouvelles concerns seem to be

tesselations of the plane
planification of polyhedra
first steps toward star polyhedra
mysticism and magic.

Actually mysticism is theoretical, whereas magic is a τέχνη because it proposes concrete realizations. Still, its occult nature makes it inappropriate for explicit treatment in an introduction and, thus, Bouvelles contents himself with a quick presentation of some ancient mystical and magical symbols such as the star pentagon (pentagram) and the star hexagon (hexagram), geometrical correlates of the star figurate numbers. In fact, he also includes other themes in his book that are of a more theoretical nature. Speaking of the area of the circle, for example, he sets out a sequence of concentric circles with integral-valued radii – which, of course, mirrors the sequences of centered figurate polygons – and observes that the proportion of their several areas is as that of the square numbers; indeed, his constant preoccupation with ratios and proportions harkens back to "sacred arithmetic".

The Egredient Pentagon

Egredient angles of a figure are its external angles. Thus an egredient pentagon is a star pentagon formed by producing the sides of a regular pentagon and thereby creating a five-pointed star. Bouvelles consistently uses the phrase "*pentagone ſaillant ou egredient*", but tends to drop the *ſaillant* thereafter. He actually starts, however, by observing that the diagonals of the pentagon create another, smaller pentagon in the original's interior, though inversely situated (*contrepoſé*). This is an important observation because it shows the existence of incommensurable quantities (see von Fritz (1945)) and was reportedly used as a Pythagorean emblem (see, Fossa (2006)). Bouvelles does not mention this, as behooves his introductory purposes, but he certainly was quite aware of the fact. He also does not explicitly state that the star pentagon formed by the diagonals is an egredient pentagon, but that such is the case is entirely obvious since, in Figure 10, $\overline{\mathrm{GFC}}$, for example, is, by construction, a segment and, therefore, $\overline{\mathrm{GC}}$ is $\overline{GF}$ produced. Hence, the egredient pentagon in Figure 11 will produce Figure 10 by connecting the successive points of the star.

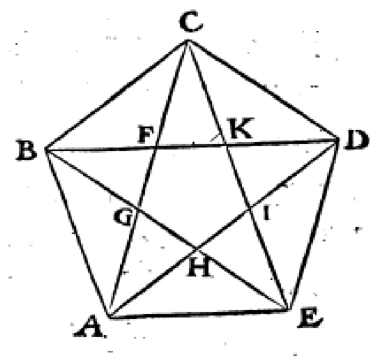

Figure 10. Diagonals of a pentagon.
Source: Bouillus (1542), p. 22a.

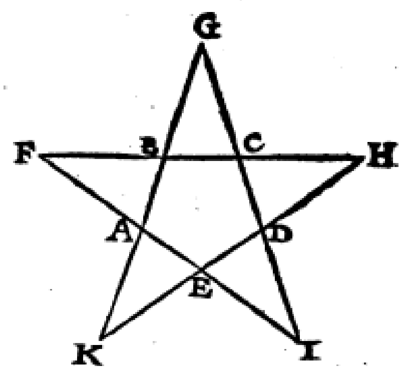

Figure 11. An egredient pentagon.
Source: Bouillus (1542), p. 22b.

The only proposition that Bouvelles advances about the egredient pentagon is that the sum of its angles (that is, the angles at the star points F, G, H, I and K in Figure 11) is two right angles. Earlier, Bouvelles had shown that the triangle ACE in Figure 10 is isosceles, each of whose base angles (the angle at A and the angle at E) is double of its third angle (the angle at C). From this, it is almost immediate that the arm of the star divides the angle of the larger pentagon into three equal parts. Thus, the interior angles of this larger pentagon consists of 15 angles all equal to the angle at C of the triangle ACE. But he had also shown that the interior angles, all together, "are worth" (*valent*) six right angles. Only five of these 15 angles, however, belong to the star. Hence, since one third of the 15 angles belong to the star, they will be worth one third of the six right angles, that is, two right angles.

Now, it is not at all clear why Bouvelles goes out of his way to prove this seemingly obtruse proposition, for he tells us nothing more about the egredient pentagon and just merrily moves on to consider the hexagon. Before following him to the next polygon, however, we may observe a singular import of the present proposition for the artisan. Since the five angles at the star points comprise two right angles, we can adjoin them adjacently so that the star points come together at a single point. The result will fill out a straight angle and, in fact, the resulting figure will be half a regular decagon. Thus, Bouvelles has in fact bestowed on the artisan an easy way to construct the regular decagon and, consequently, a ten-pointed star. (He does not teach his readers how to bisect the angle.) In any case, he is also very interested in, fairly systematically, giving relative measures for the component triangles of the polygonal figures, which would presumably be of use to the artisan in planing his compositions.

The Egredient Hexagon

The hexagon has two kinds of diagonals. The longer of them pass through the center of the hexagon. The shorter, when they are all drawn in, form a star hexagon in the interior of the original hexagon. As in the case of the pentagon, by construction, the star will be an egredient hexagon. The area of the smaller hexagon, GHIKLM in Figure 12, will be one third that of the original hexagon ABCDEF. Bovelles shows this by decomposing the figure into component parts, just as figurate numbers are investigated by decomposition (see Figure 5). His proof consists in observing that the larger hexagon has 18 component triangles, whereas the smaller one has but six. Since all of the component triangles have the same area, the stated ratio is established. As is typical of Bouvelles' exposition, this is a rather abreviated proof, especially taking into account his intended readership, the major problem being that of showing, for example, that the area of triangle ABG is equal to that of triangle AGM. To do so, one would draw a parallel to $\overline{BF}$ through A and observe that the two aforementioned triangles, contained in the two parallels, have equal bases. This proves that they have equal areas by the appropriate proposition that he had already proven earlier (p. 14b).

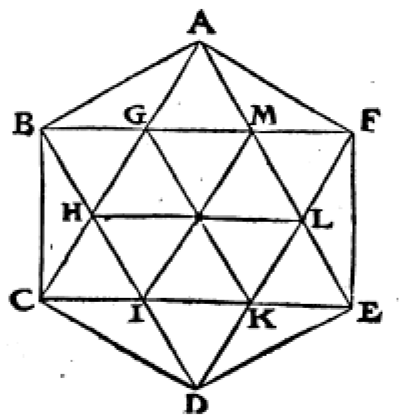

Figure 12. Some diagonals of a hexagon.
Source: Bouillus (1542), p. 23b.

It is also evident from Figure 12 that the area of the egredient hexagon is double that of the hexagon of which it is a star. Since Bouvelles, for expository reasons, does not identify the star in the figure as an egredient hexagon, he has to prove this later on (p. 25b), which he does in exactly the same way, by decomposition. The result parallels one which we have already seen for the figurate centered star hexagon with the central stone removed.

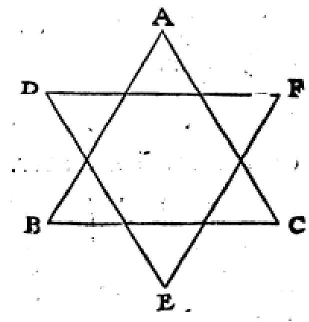

Figure 13. Egredient hexagon as two triangles.
Source: Bouillus (1542), p. 25a.

Bouvelles next observes that the angles of the egredient hexagon – those at the six star points – are equal to four right angles. This is because the star arms are equilateral and, therefore, equiangular triangles, so that three of these angles make up two right angles and, hence, six of them make up four right angles. So this is an easy observation, equivalent to that acording to which the hexagon is composed of six equilateral triangles (p. 23a).

In terms of its composition, Bouvelles observes that the egredient hexagon can also be seen as being formed from two equilateral triangles. They are to be situated inversely to each other in such a way that each of the triangles cuts the sides of the other in three equal parts, as triangles ABC and triangle DEF in Figure 13.

The Egredient Heptagons

The regular heptagon has three kinds of diagonals. The longest contains the center of the hexagon and divides it into two equal parts. Since they all meet at the center, they do not form egredient heptagons.[10] The shortest of these diagonals, when all drawn in, analogously to the previous cases, form a regular heptagon in the interior of the original one. The same happens when the intermediate sized diagonals are all drawn in. Bouvelles, however, breaking with his procedure in the two previous cases, does not draw in the diagonals and advances directly to the egredient heptagons by producing the sides.

[10] Actually, it would not be remiss, from a Pythagorean arithmetical standpoint, to consider the center point of any regular polygon as a polygon of that type. Hence, given any regular *n*-gon, we would obtain an *n*-pointed star, whose star arms would be, instead of triangles, the segments from the center to the vertices. In the case of the hexagon, for example, the resulting star would be the familiar asterisk (*). Since Bouvelles gives us no leave for such speculations, however, we do not follow up on this thought here.

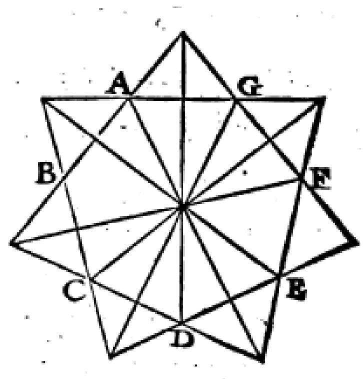

Figure 14. An egredient heptagon.
Source: Bouillus (1542), p. 27a.

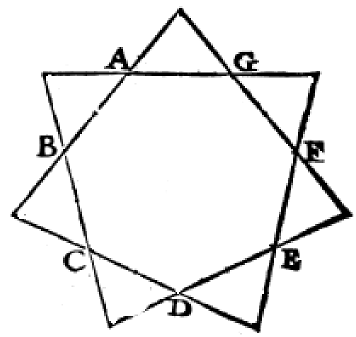

Figure 15. Previous figure simplified.

Bouvelles considers first the egredient heptagon created by producing the sides adjacent to each side of the original heptagon. As in the case of the egredient pentagon and the egredient hexagon, the base of the star's arms are the sides of the original heptagon (see Figure 15). He has but little more to say about this star, except for observing that a straight line from any star point passing through the center of the original heptagon will contain the opposite vertex of the original heptagon and divide it, and thereby also the egredient heptagon, into two "equal" (*eſgalles*) parts. Such is, for example, the segment from the topmost star point in Figure 14 to the vertex labelled D. Observe that in Figure 14 only six of the seven bisectors are drawn in.

In order to fashion the second type of egredient heptagon (having star points H, I, K, L, M, N and O in Figure 16), Bouvelles eschews returning to the original heptagon and, rather, prolongs the sides of the first type of egredient heptagon (having star points A, B, C, D, E, F and G). That is, he produces the sides of the star arms of the first type of egredient heptagon, as is shown in Figure 16 (in which I have highlighted the first star to make its structure clearer). Since the star arms are larger and sharper (*plus aguz*), he calls this star "much more egredient" (*moult plus egredient*) than the former one. Figure 16 also makes clear that this procedure is equivalent to producing the sides adjacent to each angle[11] (as opposed to the sides adjacent to each side[12]) of the original heptagon.

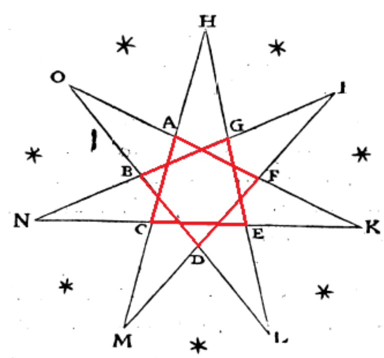

Figure 16. Another egredient heptagon.
Source: Bouillus (1542), p. 27b, modified.

[11] Schläfli symbol 7/3.
[12] Schläfli symbol 7/2.

Finally, Bouvelles observes that the sum of the angles at the star points (*angles exterieurs*) of the sharper star is equal to two right angles. Recalling the corresponding result for the pentagram, he declares (Bouillus, 1542, p. 27b) that "seven are worth as much as five" (*Sept valét dóc autant que cinq.*) or, equivalently, the point angle of the pentagonal star is to the point angle of the sharp heptagonal star as seven is to five. Although Bouvelles does not tell the reader what happens in the other case, it is easy to calculate that its point angle is thrice that of the sharper star.

He continues to make observations about the relative proportions of the regular polygons, but does not extend his further investigations to the polygonal stars.

Other Kinds of Stars

As Bouvelles himself (Bouillus (1542, p. 22b)) observed, there can be no egredient triangle or square because the prolongations of the appropriate sides, in the case of the triangle, already intersect at the vertecies of the triangle, or, in the case of the square, do not intersect. Other kinds of stars can, of course, be produced artificially by merely appending isosceles triangles to the sides of a polygon.[13] Bouvelles, although he does not call them "stars" (but neither does he call egredient polygons "stars"), presents figures of this type in the context of the planification of solid angles.

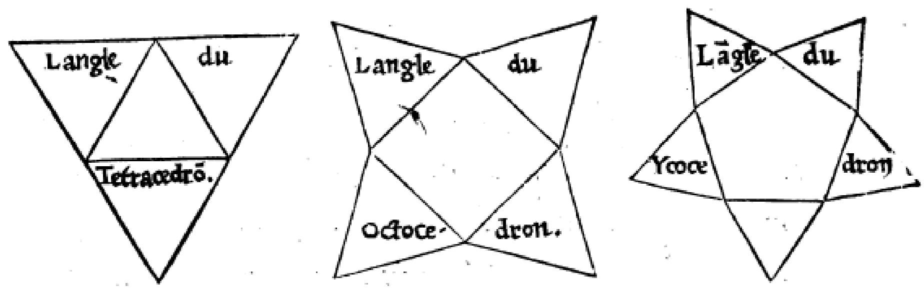

Figure 17. Three stars with equilateral arms.
Source: Bouillus (1542), p. 39a.

Figure 17 shows three such figures resulting from placing of equlateral arms along the sides of a "middle" (*moyen*) polygon. When the star points are lifted up and meet at a single point above the middle polygon, we obtain a solid angle, whose base is the aforesaid middle polygon. Thus, for the middle equilateral triangle, we obtain the angle of the regular tetrahedron; for the middle square, we obtain the angle of the regular octahedron and, for the middle regular pentagon, we obtain the angle of the regular icosahedron. Bouvelles then observes that, upon using the regular hexagon as the middle polygon, no solid angle is formed because the arms of the star do not meet above the proposed base, but fall back down on it. In fact, the star is, in this case, none other than the egredient hexagon.

[13] We have already seen a star point (note 10). A star segment would be a rhombus.

Although Bouvelles does not mention it, another interesting case (which I only mention in passing) for the artisan would be stars in which the arms are not all congruent. Such, for example, is the familiar star representing the cardinal points of the compass, in which the main points (N, S, E, W) have the longest arms, intermediate directions (NE, SE, SW, NW) have shorter arms and the next subdivision (NNE, ENE, ESE, SSE, SSW, WSW, WNW, NNW) has even smaller arms. Stars of this kind are often refered to as "roses".

If we be willing to countance another degree of abstraction, we could have "stars" whose arms are not triangles. Bouvelles, still in the context of solid geometry, puts four squares about a middle square and five regular pentagons about a regular pentagon (see Figure 18). In the first of these cases, lifting up the arms, but not to a common point, produces a cube[14] and, in the second, half of a dodecahedron.

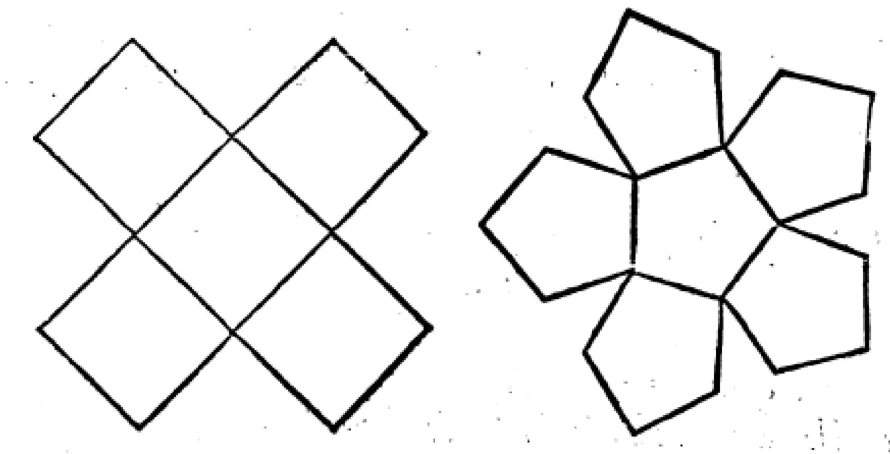

Figure 18. A square star with square arms and a pentagonal star with pentagonal arms.
Source: Bouillus (1542), p. 39b.

Another step would be to accept irregular stars, so that, for example, the Pythorean Theorem (Bouillus,1542, p. 16b) could be subsumed under the concept of "star". Hence, although, as already stated, Bouvelles does not use the word "star" in his text, the star concept does provide us with a unifying theme for many of the topics treated of in the *Livre singulier*.

[14] The figure is not a true planification of the cube because it lacks one side thereof.

Conclusion

Charles de Bouvelles was not the first writer to investigate star figures mathematically. Both Thomas Bradwardine (1295?–1349) and Adelard of Bath (1075–1160), for example, had already done so. Brigaglia, Palladino & Vaccaro (2018), in this sense, characterize his work as derivative, although they also point out that his proofs by decomposition into component triangles is noteworthy. We should, however, take into account that the *Livre singulier* was conceived of as an introductory work. Even so, the topic of star polygons was not standard in geometrical texts and thus it is interesting to see how much of what Bouvelles presents in his text can be tied together, albeit loosely and incipiantly, by star figures.

It is also significant that the study of star polygons naturally lends itself to description by ratios and proportions. This is, of course, important in a time in which the concept of function has not come into its own, so that equations were set up largely through consideration of proportions. The concept of proportion, however, takes on a special importance for Bouvelles due to his Pythagorean proclivities. Indeed, the whole worldview of the Pythagoreans – and those who were influenced by them – was imbued with the concept of ratio and proportion owing to their doctrine of the resemblance between the microcosm and the macrocosm. Bouvelles' younger contemporary Andrea Palladio (1508–1580), for example, as well as the seminal Roman architect Marcus Vitruvius Pollio (1st century BC), desired to construct buildings reflecting the mathematical structure of the universe and, of course, Plato's *Republic* is explicitly structured by comparing the macrocosm (the universe), the mesocosm (the State) and the microcosm (man), although much of his exposition is couched in quantitative, not qualitative, terms.

As a Pythagorean, Bouvelles believes that arithmetic is superior to geometry and, in this regard, it is probable that he saw many of his geometric results as being akin to those found in his study of figurate numbers. Even his noteworthy method of decomposition of geometric figures into triangles may have been inspired by similar considerations with regard to the investigation of such figurate numbers. As a Christian, Bouvelles would be interested in incorporating the esoteric Pythagorean mathematical insights into Christian doctrine. Hence, one of the important ends of the *Livre singulier* would be to prepare the artisan for graphic design.

Graphic design is not only important because of the symbol's esoteric meaning for mystical knowelege among the elite and ocult knowelege for magical praticioners. It is also a method for channeling God's grace to those ignorant of deeper philosophical questions. Stained-glass windows, such as the one he gifted to the *Collégiale* at Saint-Quentin, were supposed to perform such a function for the brethren who were illiterate and/or had no Latin with which to participate more fully in the Mass. The principle goes back at least to Plato's espousal of astrology based on the "music of the spheres": each of the moving planets emits a characteristic musical note, which is implanted on the soul of the newborn when the planet is overhead at the time of the child´s birth. Thus, the esoteric function of graphic design for Bouvelles, as for many of his contemporaries, was related to Christian charity and the propagation of the faith.

References

BOUUELLES, Charles de. 1542. *Liure ſingulier & vtile, tovchant l'art et practique de Geometrie*. Imprimé à Paris par Simone de Colines.

[BOUILLUS]. *Geometrie en francoys*. 1511. Imprime a paris par Henri estiêne. [Imprimé à Paris par Henri Estienne.] Disponível em https://play.google.com/books/reader?id=ws5bAAAAQAAJ&pg=GBS.PP6&hl=en_US.

BOUILLUS (Samarobrinus), Carolus. 1510. *Liber de perfectis numeris*. (Escrito em 1509.) In: BOUILLUS (Samarobrinus), Carolus. Sem título. [*Que hoc volumine continentur*.] 1510. Paris. Disponível em https://archive.org/details/bub_gb_MumicQwuD8gC/page/n371/mode/1up.

BOUILLUS (Samarobrinus), Carolus. 1510a. *Libellus de mathematicis roſis*. In: BOUILLUS (Samarobrinus), Carolus. Sem título. [*Que hoc volumine continentur*.] 1510. Disponível em https://archive.org/details/bub_gb_MumicQwuD8gC/page/n387/mode/1up.

BRIGAGLIA, Aldo, PALLADINO, Nicla & VACCARO, Maria Alessandra. 2018. Historical Notes on Star Geometry in Mathematics, Art and Nature. In M. Emmer & M. Abate. (eds) *Imagine Math 6*. Springer. Disponível em: Cham. https://doi.org/10.1007/978-3-319-93949-0_17.

DE MORGAN, Augustus. 1915. *A Budget of Paradoxes*. Chicago: Open Court. [Originally 1872.]

FOSSA, John A. 2021. Maurolico e Pascal: Indução Matemática e Demonstração por Exemplificação. *Revista Brasileira de História da Matemática*. Vol. 21, n. 41, p. 11–24.

FOSSA, John A. 2020. Sobre a proposição de que todo número perfeito é um número triangular. *Revista Brasileira de História da Matemática*. Vol. 20, n. 40, p. 32–45.

FOSSA, John A. 2006. On the Pentagram as a Pythagorean Emblem. *Revista Brasileira de História da Matemática*, v. 6, n. 12, p. 127–137.

HØYRUP, Jens. 2002. *Lengths, Widths, Surfaces: A Portrait of Old Babylonian Algebra and its Kin*. New York: Springer.

MIMESI. 2012. Charles de Bouvelles, philosophe et pédagogue. Departament de Filologia Catalana, Universitat de Barcelona. Disponível em https://stel2.ub.edu/mimesi/agenda/charles-de-bovelles-philosophe-et-p%C3%A9dagogue.

O'CONNOR, J. J, e ROBERTSON, E. F. 2002. Charles de Bouvelles. Disponível em: https://mathshistory.st-andrews.ac.u k/Biographies/Bouvelles/.

OOSTERHOFF, Richard J. 2017. 'Secrets of Industry' for 'Common Men': Charles de Bovelles and Early French Readerships of Technical Print. In: K. A. E. Enenkel, N. Hodson & S. Fransen, Eds. *Translating Early Modern Science*. Leiden: Brill. Disponível em: https://brill.com/edcollbook-oa/title/35095.

SAITO, Fumikazu. 2017. Algumas breves considerações sobre os tratados de geometria prática publicados no contexto do "saber-fazer" matemático quinhentista. Marcos Vieira Teixeira & Mariana Feiteiro Cavalari, Eds. *Anais do XII Seminário Nacional de História da Matemática*. Sociedade Brasileira de História da Matemática. Disponível em: https://www.sbhmat.org/download/download?ID_DOWNLOAD=6.

VON FRITZ, Kurt. 1945. The Discovery of Incommensurability by Hippasus of Metapontum. *Annals of Mathematics*, v. 48, n. 2, p. 242–264, 1945.

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