LUCA PACIOLI AND THE "CONTROVERSY OF THE PERSPECTIVE": THE CLASSIFICATION OF THE MATHEMATICS FROM THE CLASSICAL ANTIQUITY TO THE END OF THE QUATTROCENTO.

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There have been several classifications of sciences toward the history. Until now, a demarcation criterion which allows discerning what should or should not be called science and how to distinguish it related to its nature has been debated.

In the medieval world, the branches of knowledge, responsible for the individual educational basis, were called *Artes Liberales* (Liberal Arts). These ones were cultivated for the education of the free man (lat. *Liber*) opposing to the *Artes Iliberales* (Illiberal Arts) which had economic purposes.

Liberal Arts can be divided into two groups: the *Trivium* (or *Artes Sermocinales* or *Triviales*) and the *Quadrivium* (or *Artes Reales* or *Physicae*, as well *Quadriviales*). The *Trivium*, which means "three-way crossing", is constituted of the Grammar, Rhetoric and Dialectic (or Logic). Such arts were considered elementary ones. The mathematical subjects Arithmetic, Geometry, Astronomy and Music compounded the *Quadrivium* which means "four-way crossing". The arts of the *Quadrivium* were considered intermediaries, having as a purpose the acquisition of a high level of knowledge through the Philosophy and the Theology. Having been well known, the mnemonic verses of medieval circulation summarize the Liberal Arts functions:

> *Gram loquitur, Dia verba docet, Rhet verba colorat,*
> *Mus canit, Ar numerat, Geo ponderat, Ast colit astra*
> (LEWIS,1994, p. 186). [1]

Certainly, any classification of the branches of knowledge, besides its huge influence, could not be accepted unanimously. Frà Luca Pacioli (1445 – 1517?), in the epicenter of the advent of the Linear Perspective, supported the inclusion of this one in the arts of the *Quadrivium*:

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[1] *"The Grammar speaks, the Dialectic teaches the words, the Rhetoric colors the words, the Music sings, the Arithmetic counts, the Geometry weights, the Astronomy tends the planets."*

However, our wisdom, even low and unable, converted them into three or five, it means, Arithmetic, Geometry and Astronomy, excluding from these ones the Music, due to as many reasons as they give to exclude from the five ones the Perspective, or adding this one to the four others, as so many reasons as the ones which added to the three ones the Music. […] I suppose that many wise men should not be wrong, therefore, besides their statements, my ignorance does not surrender (PACIOLI, 1498, f.VIIIIv – Xr).[2]

The words of Pacioli, a mixture of intellectual stubbornness and Franciscan humbleness, introduce us to a debate concerned to the status of the Perspective and the Painting at the end of the *Quattrocento*, having also the participation of his friend Leonardo Da Vinci (1452 – 1519). Among several considerations to be done in the study of the Renaissance, two important points can not be put aside - the back to the Antiquity and the development of the Perspective as the interpretation of the reality.

The goal of this study is to introduce the discussion of Luca Pacioli about the importance of the Perspective as a mathematical subject. We start with a short history of the *Quadrivium*, its origin, development and establishment.

1 - Classical Antiquity

Classical Antiquity is a broad term for a long period of cultural history, the Greco – Roman civilization, between VI BC and V AC. The Greek world developed a cultural model of intellectual reflection and it was taken by the Romans, influencing, in a general point of view, the occidental civilization.

According to the classicist Werner Jaeger *"the higher we judge the artistic, religious and political realizations from the ancient people, the history, which we can consciously call culture, only begins with the Greeks"* [3] (JAEGER, 1979, p.4). Bertrand Russel states *"Philosophy and science as we know them are Greek inventions. The rise of Greek civilisation which produced this outburst of intellectual activity is one of the most spectacular events in history. Nothing like it has ever occurred before or since".* (RUSSELL, 2003, p. 20). We will not discuss such statements, but they give evidence to the importance given to the intellectual developments made by the Greeks according to a considerable number of authors.

Next, we will study briefly some terms used by the Greeks to define the knowledge types related to their Mathematics.

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[2] *"Ma el nostro iudicio benche imbecille et basso sia o tre o cinque ne constringe. cioe Arithmetica. Geometria. e astronomia excludendo la musica da dicte per tante ragioni quante loro dale .5. La prospectiua e per tante ragione quella agiognendo ale dicte quatro per quante quelli ale dicte nostre .3. la musica. [...] pur existimo tanti saui non errare. E per lor dicti la mia ignoranza non si suelle."*

[3] *"So hoch wir auch die künstlerische, religiöse und politische Bedeutung der früheren Völker schätzen mögen, beginnt doch die Geschichte dessen, was wir als Kultur in unserem bewussten Sinne bezeichnen können, nicht eher als bei den Griechen."* (JAEGER, 1973, p. 3).

1.1 - `Tevcnh kaiÆ ejpisthvmh` (*Téchne* and *Epistéme*)

The Greek word `tevcnh` (*téchne*) is used to be translated as "art", but, among other acceptations, we could emphasize "craft", "industry", "theoretical knowledge" and "method". *Téchne* defines handle ability or ability of the spirit, a branch of the knowledge, a practical science. `jEpisthvmh` (*Epistéme*), could also be translated as "art" or "skill", "knowledge" or "science". If *téchne* is the practical science, *epistéme* is a theoretical science, the true knowledge, against the thoughtless opinion (`dovxa`) (cf. PLATO, *Republica* V, 477b). As it is known, having a sharp definition of such terms is quite difficult, once the semantic depends on the period studied, the considered author and the evolution of his thought. There is a close relation and also a fundamental contrast between *epistéme* and *téchne*. Either they are used without any distinction or as an opposite meaning (cf. PARRY, 2003).

Aristotle distinguishes clearly the *epistémai* and the *téchnai* in his Nichomachean Ethics, even such distinction has not always been observed on the whole of his work. Together with the `frovnhsi"` (*phrónesis*, prudence), the `sofiva` (*sophia*, knowledge) and the `nou'"` (*noûs*, pure reason), other activities derived from the soul rationality, are the ones called intellectual virtues. The *téchnai* are closer to the experience, they do not focus on the knowledge by itself, and they are activities about what it is non-necessary. They are concerned to the reproduction of empirically verifiable knowledge, without looking for explanations, it means, the *téchnai* are related to the production (`poivhsi"`, *poiésis*), being neither in itself nor by itself an end. The *epistémai* are concerned to the universal knowledge, of the necessary, of the absolute, they search for the cause to understand better and work with the demonstration.

On the whole, there had been some identification between science and philosophy for the Greeks. Therefore, when the division of the sciences is considered, on the Hellenic culture, the division of the philosophy is taken as well.

1.2 – `Maqhmatikhv` (*Mathematiké*) and the origin of the *Quadrivium*

The Greek word `maqhvmata` (*mathémata*), which could be translated as "Mathematics", is the plural of `mavqhma` (*máthema*), which could be translated as "study", "science" or "knowledge". These words are related to the verb `manqavnw` (*mantháno,* "to learn", "to study", "to understand") and to `maqhmatikov"` (*mathematikós*, "dedicated to the study"). In Plato, the term *máthema* is used in a much broader meaning; it refers to any object of study or instruction. According to Sir Thomas Heath, *"the words `maqhvmata` and `maqhmatikov"` do not appear to have been definitely appropriated to the special meaning of mathematics and mathematicians or things mathematical until Aristotle's time"* (HEATH, 1981, p.10).

In a fragment attributed to Archytas of Tarento (c.428 – c. 347 AD), the philosopher king friend of Plato, the use of the term *mathémata* in the meaning of the mathematical sciences is found (cf. entry `mavqhma` in LIDDELL, 1940):

> Let us now cite the words of Archytas the Pythagorean, whose writings are said to be mainly authentic. In his book On Mathematics right at the beginning of the argument he writes thus:

> "The mathematicians seem to me to have arrived at true knowledge, and it is not surprising that they rightly conceive the nature of each individual thing; for, having reached true knowledge about the nature of the universe as a whole, they were bound to see in its true light the nature of the parts as well. Thus they have handed down to us clear knowledge about the speed of the stars, and their risings and settings, and about geometry, arithmetic and sphaeric, and, not least, about music; for these studies [maqhvmata] appear to be sisters" (THOMAS, 1991, p. 5).[4]

In this passage, named Fragment 1 (Frag. 1), Archytas lists four sciences (*mathémata*), Geometry, Arithmetic, Astronomy (Sphaeric) and Music, configuring in such way, the most ancient testimony of the existence of a pythagorical *Quadrivium*.[5] As it will be seen, the formation program of the philosopher presented in the Republic of Plato reflects the classification of the *mathémata* of Archytas. In such work, Socrates discusses about the Astronomy and the Harmony as sisters, in an explicit reference to the Phytagoreans (PLATO, *Republica*, VII, 530d).

Saint Anatolius of Alexandria (III century AD), assures that the Pythagoreans were the first to use the term *maqhmatikhv* (*mathematiké*, feminine of *mathematikós*), exclusively to the Geometry and Arithmetic:[6]

> Why is mathematics [maqhmatikhv] so named?
> The Peripatetics say that rhetoric and poetry and the whole of popular music can be understood without any course of instruction, but no one can acquire knowledge of the subjects called by the name of mathematics unless he has first gone through a course of instruction in them; and for this reason the study of these subjects was called mathematics. The Pythagoreans are said to have given the special name mathematics [maqhmatikhv] only to geometry and arithmetic; previously each had been called by its separate name, and there was no name common to both (THOMAS, 1991, p. 3).

It seems reasonable that the use of *mathematiké*, referring to the mathematical sciences, is attributed to the school of Pythagoras, because as Porphyry (c. 234 – c. 305 AD) and Iamblichus state (c. 245 – c. 325 AD), his disciples were divided into two groups: the *maqhmatikoiv* (*mathematikoí*), who learned a much more elaborated doctrine, and the *ajkousmatikoiv* (*akousmatikoí*, derived from *jakouvw*, "listen"), who were

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[4] Cited by Porphyry in his commentary about the *Harmonic* of Ptolemy (MULLACH, 1860, p. 564). About the authenticity and variant forms of the Frag.1 v. HUFFMAN, 1985 and for more details of Archytas and his writing v. HUFFMAN, 2004 e HUFFMAN, 2005.

[5] Sphaeric is identified with astronomy (cf. HEATH, 1981, p. 11 e HUFFMAN, 2004 p. 243).

[6] Saint Anatolius was bishop of Laodicéia, in Syria, around 283 AD. He is cited by Eusebius de Caesarea: *"Eusebius, who had come from the city of Alexandria, ruled the parishes of Laodicea after Socrates. [...] Anatolius was appointed his successor; one good man, as they say, following another. He also was an Alexandrian by birth. In learning and skill in Greek philosophy, such as arithmetic and geometry, astronomy, and dialectics in general, as well as in the theory of physics, he stood first among the ablest men of our time, and he was also at the head in rhetorical science. It is reported that for this reason he was requested by the citizens of Alexandria to establish there a school of Aristotelian philosophy"* (EUSEBIUS, 1890, p. 318, *Hist. Eccl.*, VII, 32). The presented quotation is found in *Definitiones* of Heron of Alexandria (c. 10 – c. 75 AD), who lived two centuries before to Anatolius! For more details v. HEIBERG, 1914 and TANNERY, 1887, p. 177.

exoteric disciples, they could just listen the Pythagoras's teachings, without seeing him (cf. PORPHYRY, 1816, p. 68; IAMBLICHUS, 1989, p. 35; MCKIRAHAN, 1994, p. 89 - 91). Other testimony, a little later, of this Pythagorean classification, as well the existence of the other classifications of the Mathematics, can be found on the work of Proclus (412 – 485 AD). A passage of his Commentary to the Book I of the Elements of Euclid has been cited:

> The Pythagoreans considered all mathematical sciences to be divided into four parts: one half they marked off as concerned with quantity (*posovn*), the other half with magnitude (*phlivkon*); and each of these they posited as twofold. A quantity can be considered in regard to its character by itself or in its relation to another quantity, magnitudes as either stationary or in motion. Arithmetic, then, studies quantity as such, music the relations between quantities, geometry magnitude at rest, spherics magnitude inherently moving (PROCLUS, 1992, p. 29 - 30).
> But others, like Geminus, think that mathematics should be divided differently [...] (PROCLUS, 1992, p. 31).

According to Proclus, the stoic Geminus (c. 10 BC – c. 60 AD) considers, in his division of the Mathematics, on one hand, the sciences related to the intelligible things, Arithmetic and Geometry and, on another hand, the ones related to the sensitive things, Mechanics, Astronomy, Optics, Geodesy, Canonics and Logistics (cf. TANNERY, 1887, p. 38 – 52). Anatolius makes the same classification (cf. THOMAS, 1991, p. 19 and TANNERY, 1887 p. 42 – 43).

1.3 - Plato

Plato, in his work *Statesman*, divides the science (*epistéme*) in *praktikhv* (*praktiké*), which means the practice or science of the action, as the architecture, and *gnwstikhv* (*gnostiké*), which is the science of the knowledge or theoretical, as the Arithmetic.[7] We could consider it as his division of the science. However, Plato does not show in his writings, in a explicit way, a division of the Philosophy and from some old testimonies, his system can be divided in three parts: Dialectic, i.e. the science of Idea by itself; Physics, i.e. the knowledge of the Idea incorporated in the world of the phenomenona, and the Ethic, i.e. the science of the Idea incorporated in the human behavior in the society (TURNER, 1911; PECK, 1898; SCHWGLER, 1856, p.82-83). According to Plato, the Mathematics was part of the propaedeutics to the Philosophy.

"Let no one unversed in geometry come under my roof".[8] It is said about this famous statement written over the entrance of the Academy of Plato. Such requirement illustrated the great importance gave by Plato to the Mathematics, especially to the Geometry, as "God

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[7] "*tauvth/ toivnun sumpavsa" ejpisthvma" diaivrei, th;n me;n praktikh;n proseipwvn, th;n de; movnon gnwstikhvn*" ["*In this way, then, divide all science in two parts, calling the one practical, and the other purely intellectual*"] (*Politicus*, 258e).

[8] According to the Byzantine writer Johannes Tzetzes (c. 1110 – c.1180), "*Pro; tw'n proquvrwn tw'n auJtou' gravya" uJph'rce Plavtwn* 'Mhdei;" ajgewmevtrhto" eijsivtw mou th;n stevghn'" ["*Over his front doors Plato wrote: 'Let no one unversed in geometry come under my roof'*"] (THOMAS, 1991, p. 386 - 387). It has been cited in a brief version: "*ajgewmevtrhto" mhdei" eijsivtw*".

always geometrizes".[9] It is noticed in the *curriculum* of philosopher kings (Guardians), proposed by Plato in the VII Book of the Republic, the fundamental role of the Mathematics. The goal of his program was to prepare the spirit to the Dialectic, whose end objective is the knowledge of the Good (cf. 533 b-e). The future government should have a sharp knowledge of the Mathematics, which above its usefulness, in the war for instance, would make easy the passage from the mutability of the soul to the true and to the essence (cf. 525c), reviving an organ, whose salvation is much more important than ten thousand of organs of the vision (cf. 527e). This is the sequence of study (*mathemáta*) which the Guardians, around twenty and thirty years old, should dedicate two years after of Music and Gymnastic graduation (II, 376e): Arithmetic (522c)[10], Geometry (526c), Stereometry (528a), Astronomy (528e) and Harmonics[11] (530d). We have here the same components (*téchnai*) from the ulterior one called *Quadrivium*, adding the Stereometry.

Considering that the solid geometry had already been studied by the Pythagoreans, by Democritus (c. 460 a.C. – c. 370 a.C.) and others, the distinction between the Geometry and the Stereometry only a formality, to show the small advancement made, in the epoch, in this "new science" (cf. HEATH, 1981, p. 12). In fact, we can verify the incorporation of Stereometry to Geometry, made by Plato in his work *Laws* (VII, 817e):

> Then there are, of course, still three subjects [*trivva maqhvvmata*] for the freeborn to study. Calculations and the theory of numbers form one subject; the measurement of length and surface and depth make a second; and the third is true relation of the movement of the stars one to another (THOMAS, 1991, p. 21).

Besides corroborate with the veracity of the conclusion about Geometry as the science dealing with the plane figures and solids, this passage is more one example which can reinforce the use of the word *mathémata* as it was discussed in the section 1.2. According to Heath, the pre-eminence given to the Mathematics, in the Platonic Educational Scheme, can have encouraged the habits of treating them as *mathémata* (HEATH, 1981, p. 10). It is also observed the particularity of such matters being explicitly classified as free man study objects, concurring with the conception of the liberal arts which had already been mentioned.

According to Jaeger, the sophists were the first ones who had included the *Quadrivium*, on the highest level of the Greek culture (JAEGER, 1979, p. 341). It is difficult to know in what way Plato received them, what we really know is that others had already shown as fundamentals of the education.[12] Protagoras, in the dialogue of written by Plato, exposes the education proposals by other sophists, against its based on the politic art, to bring up good citizens:

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[9] "*ajei; Qeo;" gewmetrei'*" (cf. THOMAS, 1991, p. 387; PLUTARCH, *Convivalium Disputationem,* VIII, 2).

[10] *Logistikhv* ("calculation") *kai; ajriqmhtikhv* ("number theory"). Cf. HEATH, 1981, p. 13.

[11] Platão uses the term *aJrmoniva* (*harmonia*) in contrast with *mousikhv* (*mousiké*) as popular music from the lyre masters (cf. THOMAS, 1991, p. 7).

[12] Cf. *Hippias Major*, 285b; *Theaetetus*, 145a-d. About Plato's contact with the Pythagoreans v. CICERO, 1877, p. 25, *Tusculanae Disputationes*, I, 17.

> For Hippocrates, if he comes to me, will not be treated as he would have been if he had joined the classes of an ordinary sophist. The generality of them maltreat the young; for when they have escaped from the arts [thvcnai] they bring them back against their will and force them into arts [thvcnai], teaching them calculation [logismovv"], astronomy and geometry and music (PLATO, Protagoras, 318d-e).

The Scottish classicist James Adam considers this passage as a record of the use of the term "art" (*téchne*) applied *par* excellence to the *Quadrivium*, in Plato's time. According to him, the Plato's propaedeutic art, shown in the *Republic*, are essentially the same as the medieval *Quadrivium* (ADAM, 1901, p. 220).

1.4 - Aristotle

"*All men by nature desire to know*".[13] Aristotle begins his Metaphysics with this statement. It is through wonder that men begin to philosophize,[14] and by the natural desire of learning and the leisure, the Egyptian priests became surprised with some celestial phenomena and from their search for explanations the mathematical arts (*téchnai*) were born.[15] It is on its authority (not exclusively) that those ones who call him the Philosopher are based when they start a work, during the Middle Ages and the Renaissance.[16]

The division of the knowledge or classification of the sciences (*epistémai*) of Aristotle is constituted of three groups:[17]

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[13] "*Pavnte" a[nqrwpoi tou' eijdevnai ojrevgontai fuvsei*" (*Metaphysica*, I, 1, 980a, 1).

[14] "*diaÆ gaÆr toÆ qaumavzein oiJ a[nqrwpoi kaiÆ nu'n kaiÆ toÆ prw'ton h[rxanto filosofei'n*" ["*It is through wonder that men now begin and originally began to philosophize*"] (*Metaphysica* I, 2, 982b, 12).

[15] "*dio; peri; Ai[gupton aiJ maqhmatikai; prw'ton tevcnai sunevsthsan, ejkei' ga;r ajjfeivqh scolavzein to; tw'n iJerevwn e[qno"*" ["*Thus the mathematical arts originated in the neighborhood of Egypt, because there the priestly class was allowed leisure*"] (*Metaphysica*, I, 1, 981b, 23-24).

[16] The title "the Philosopher" was attributed to the Stagirite by the authors as St. Thomas Aquinas (cf. *Summa Theologiae*, I q. 1, a. 1, a. 3, a. 4 etc). Frequently, it is found in the beginning of the works of the medieval and renaissance authors, quotations of Aristotle (cf. "*Il Convivo*"of Dante). Such current use of quotations, mainly in fabulous and profanes works, deserved the mention of Miguel of Cervantes, in the Prologue of his book *Don Quijote de la Mancha*: " (...) *tan llenos de sentencias de Aristóteles, de Platón y de toda la caterva de filósofos, que admiran a los leyentes y tienen a sus autores por hombres leídos, eruditos y elocuentes?*". From the renaissance mathematicians, two Italians and one Portuguese can be cited. Luca Pacioli begins the Chapter II of his *De Divine Proportione*: "*Propter admirari ceperunt philosophari. Vole Ex^o D. la proposta auctorita del Maestro de color che sanno che dal uedere hauesse initio el sapere...*" (PACIOLI, 1498, *Divina Proportione*, f. IIIr). Niccolò Tartaglia, in his translation of the Elements, writes: "*Tvtti gli huomini, Magnifici e Preclarissimi Auditori, (come scriue Aristotele nel primo della Methaphisica) naturalmente desiderano di sapere*" (TARTAGLIA, 1565 , f. 3r, *Lettione de Nicolo Tartalea Brisciano, sopra tvtta la opera di Evclide Megarense, acvtissimo mathematico*). The Portuguese Gaspar Nicolas writes in his *Tratado de Pratica Darysmetica*: "*Todos hos homeēs naturalmente ylustre senhor desejam saber: segūdo aristotiles no prymeyro da metafisyca [e]t como quer que as artes liberaes ha arismetyca seja fundamento de todas...*" (NICOLAS, 1519, *Prologo*). Up to the beginning of century XII, the Aristotle thought was known basically through the works (translations, commentaries, etc.) of Boethius (480 - 524). Others of his translators which deserve special reference are Guillermo de Moerberke (1215 - 1286) and the Cardinal Giovanni Bessarione (1402 - 1472).

[17] "*w&ste eij pa'sa diavnoia h~ praktikh; h~ poihtikh; h~ qewrhtikhv* (...)" ["*Thus if every intellectual activity is either practical or productive or speculative...*] (*Metaphysica* VI, 1, 1025b, 26).

- *Poetical or productive sciences* (`poihtikaiv`, *poietikai*), which study the works of intelligence made with pre-existing materials (objects and works of arts): Poetics, Rhetoric and Logic;
- *Practical sciences* (`praktikaiv`, *praktikai*), which investigate the action of the man in his several forms: Ethics, Politics and Economics;
- *Theoretical* or *speculative sciences* (`qewrhtikaiv`, *theoretikai*), the higher ones, were concerned to the principle of the existence and to the speculation: Mathematics, Physics and First Science (Metaphysics or Theology).[18]

Aristotle establishes a hierarchy among the sciences where the speculative ones have primacy[19] and, as we can see, in his classification, the Mathematics is a speculative science.[20]

1.5 Liberal Arts

The great esteem of the Greeks for the single intellectual activities, took them to certain neglect to the handwork activities. Such contrast resulted in a classification of the knowledge completely accepted in the Antiquity, in the ones which the Romans denominated *"artes liberales"* and *"artes vulgares"*.[21] As Władysław Tatarkiewicz observes, the distinction between them came up very early, making impossible to determine its author (TATARKIEWICZ, 1963 – p. 233). We can consider that there was an equivalence of acceptation between the terms *epistéme* and *téchne* from the Greeks and the *scientia* (science) and *ars* (arts) from the Latin people, respectively (cf. LEWIS & SHORT, 1879; KRISTELLER, 1951, 498).

Galen (c. 129 – c. 216), in his work *Protrepticus*, considers the Medicine, the Rhetoric, the Music, the Geometry, the Arithmetic, the Philosophy, the Astronomy, the Literature and the Jurisprudence as "venerable arts", in contrast to the "despicable arts", which depended on the handwork. Galen says, hesitating, that the Painting and the Sculpture could also be considered as belonging to the first group (GALEN, 1930, *Protrepticus*, 14).

The oldest register of the use of "*artes liberales*" can be found in the work of Cicero (106 BC – 43 BC), particularly in *De Oratore,* where it is contrasted the arts that are worthy of the free man (*"artes quae sunt libero dignae"*) with the servile arts (*"artes serviles"*)(CICERO, 1830, p. 35, *De Oratore*, III, 16). As liberals, Cicero enumerates the

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Curiously, Diogenes Laërtius (c. 200 – c. 250) attributes such division to Plato (cf. DIOGENES LAERTIUS, 1862, p. 87). Maybe this one had been adopted in the Academy during the time of Diogenes.

[18] *"w&ste trei'" a~n ei\en filosofivai qewrhtikaiv, maqhmatikhv, fusikev, qeologikev"* [*"Hence there will be three speculative philosophies: mathematics, physics, and theology"*] (*Metaphysica*, VI, 1, 1026a, 18-19). Ptolemy, in the beginning of his *Almagesto,* confirms that the authorship of this theoretical philosophy subdivision belongs to Aristotle.

[19] *"qewrhtikai; tw'n a~llwn ejpisthmw'n aiJretwvtatai"* [*"The speculative science are to be preferred to the other sciences"*] (*Metaphysica*, VI, 1, 1026a, 23). Among the speculative sciences the theology is the primate.

[20] *"ajllÅ ejsti kai; maqhmatikh; qewrhtikhv"* [*"but the mathematics is also speculative"*] (*Metaphysica*, VI, 1, 1026a, 9).

[21] They are also called `banausikavi`,*"illiberales"* or *"sordidae"*. We can also consider that to the free man, cultivator of the liberal arts, is attributed the *"otium"* (leisure, `scolhvv` in Greek).

Geometry, the Literature, the Poetry, the Natural Science, the Ethics and the Politics, however, he does not give a complete list.

To the *liberales* and *vulgares* arts, Seneca (4 BC – 65 BC), based on Posidonius (c. 135 BC – 51 BC), added the "*artes pueriles*" assigned to the instruction, and the "*artes ludicrae*" assigned to the amusement (SENECA, 1842, p 438, *Epistolae Morales,* XIII, 3). Seneca also includes among the Liberal Arts the Medicine and denies the same status to the Painting and to the Sculpture:

> I will not be induced to admit that painters or sculptors practise a liberal art, or the other ministers of luxury (SENECA, 1842, p. 436, *Epistolae Morales*, XIII, 3).[22]

It is noticed that the Roman did not have the same admiration to the Mathematics as the Greeks, once they were more interested on the cultivation of the "*Humanitas*", in special, Grammar and Rhetoric. Another fact to observe is that in the latter Latin, *mathematicus* was used in a vulgar meaning, it meant fortune-teller, astrologer, wizard (cf. ST. AUGUSTINE, *De Genesi ad Litteram*, II, xvii, 37).

The definitive organization of the Liberal Arts was born in the work of the pagan encyclopaedist Marciano Capella (century V), even same classifications of the arts had been made before. In the two first books of his work *De Nuptiis Philologiae et Mercurii et de septem Artibus liberalibus libri novem*, Cappella shows allegorically the seven Liberal Arts as virgins to the bride Philology and, in the seven following books, he discusses particularly each one of them.

2. Middle Age and the "Seven Pillars of the Knowledge"[23]

As heirs of the knowledge theory elaborated by the ancient people, we can say that, related to the division of knowledge, the medieval authors followed two great traditions: the one we denominate platonic divides the Philosophy into Physics, Ethics and Logic, and the one we denominate aristotelic divides the Philosophy into Theoretical, Practical and Poetical.

In the *De institutione arithmetica* of Boethius (c. 480 – c. 524) we found the first use of the term "*Quadrivium*", distinguishing the Arithmetic, the Geometry, the Music and the Astronomy, as indispensable to the acquiring of the knowledge (*"sapere"*), which is at the same time an intellectual and practical knowledge:

> If the investigator lacks these four parts, it will not be able to find what is true, and without this speculation of the true nothing can be straight known […]. This is the Quadrivium (BOETHIUS, 1867, p.9, *De institutione arithmetica*, I, 1).[24]

Cassiodorus (c. 485 – c. 585), disciple and friend of Boethius, added the Liberal Arts to the monks study, in the works *Institutiones divinarum et saecularum litterarum* and *De artibus*

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[22] "[...] *non enim adducor ut in numerum liberalium artium pictores recipiam, non magis quam statuarios aut marmorarios aut ceteros luxuriae ministros*".

[23] Interesting relations can be done among the Seven Liberal Arts and the meanings of the numbers 3, 4, and 7 to the Christians, particularly with the following statement of Proverbs X, 1: *"Sapientia aedificavit sibi domum excidit columnas septem"*.

[24] *"Quibus quattuor partibus si careat inquisitor, verum invenire non possit, ac sine hac quidem speculatione veritatis nulli recte sapiendum est* [...] *Hoc igitur illud quadrivium est"*.

ac disciplinis liberalium litterarum. Saint Isidore of Seville (560 - 636) defined them, in his *Etymologiae*,[25] as it follows:

> Seven disciplines are part of the Liberal Arts. The first one is the Grammar, the knowledge of the language. The second one is the Rhetoric, which by its light and abundance of its eloquence is considered necessary mainly in the civil issues. The third one is the Dialectic, known as well as the Logic, which separates in the most subtle disputes the true and the false. The fourth one is the Arithmetic, which has the relations of the numbers and its division. The fifth one is the Music, which consists in the art of the poem and the chant. The sixth one is the Geometry, which is the measures and dimensions of the Earth. The seventh is the Astronomy, which has the rules of the stars (ISIDORE, *Etymologiae*, I, 2).[26]

Isidore affirms that, according to some authors, can be considered *ars* the things that consist of the rules and of the precepts from one art[27] and *disciplina* a complete science [28]. He also attributes to Plato and Aristotle the following distinction: there is *ars* when referred to something credible or opinionable and *disciplina*, when something is discussed with true arguments about things that can not be behaved in another way. Such definitions are found in the works of Cassiodorus, with references to other authors as Saint Augustine and Capella (cf. CASSIODORUS, 1886, *Institutiones*, II, 2, 17; II, 3, 20; GUILLAUMIN, 2007, p. 62).

Hugh of Saint Victor (1096 - 1141) also retakes such definitions in his work named *Didascalicon* (cf. HUGH OF SAINT VICTOR, *Didascalicon*, II, 1).). His innovation is in the way to add to the Philosophy some vulgar arts, denominated Mechanical arts (*mechanicae*). Here we have his division of the Philosophy and its subdivisions:[29]

- *Theoretical* (*Theorica*): Theology, Mathematics and Physics;
- *Practical* (*Practica*): Solitary (Ethics), Private (Economics) and Public (Politics);
- *Mechanical* (*Mechanica*): *Lanificium* (wool manufacture), *Armatura* (Fabrication of weapons), Navigation, Agriculture, Hunt, Medicine and *Theatrica* (Science of the Theatre);
- *Logical* (*Logica*): Grammar and *Ratione disserendi* (Theory of the Argumentation).

What Hugh denominates Mathematics is exactly the *Quadrivium* and the arts of the *Trivium* are in the subdivision of the Logic. He affirms that the Liberal Arts are like great tools from

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[25] For a study about the History of the Mathematics included in the *Etymologiae* v. NOBRE, 2005.

[26] *"Disciplinae liberalium artium septem sunt. Prima grammatica, id est loquendi peritia. Secunda rhetorica, quae propter nitorem et copiam eloquentiae suae maxime in civibus quaestionibus necessaria existimatur. Tertia dialectica cognomento logica, quae disputationibus subtilissimis vera secernit a falsis. Quarta arithmetica, quae continet numerorum causas et divisiones. Quinta musica, quae in carminibus cantibusque consistit. Sexta geometrica, quae mensuras terrae dimensionesque conplectitur. Septima astronomia, quae continet legem astrorum".*

[27] *"Ars vero dicta est, quod artis praeceptis regulisque consistat"* (ISIDORE, *Etymologiae* I, 1, 2).

[28] *"quia discitur plena"* (ISIDORE, *Etymologiae* I, 1, 1).

[29] *"Philosophia divitur in theoricam, practicam, mechanicam, logicam".* (*Didascalicon*, II, 1).

which the spirit is prepared to the way of the true philosophical knowledge and, in ancient times, nobody would deserve to be called Master if he did not get to show the knowledge of these seven sciences.[30]

We will not discuss here, but deserve attention the curricular development of the medieval schools, the rise of the universities and the Arabian contribution to the setting up or new interpretations of the *Quadrivium*.

Influenced by the Arabian interpretation of the aristotelic classification of the knowledge, from century XII, some European authors, began to accept the mechanic arts as applications of the theoretical ones (cf. WHITNEY, 1990, p. 131).

3. Perspectives

A historiography tradition established that the Linear Perspective was developed in Florence in the beginning of the *Quattrocento* by Fillipo Brunelleschi (1377 - 1446).[31] At the end of the 50's, in the past century, the art historians proposed new hypotheses about the existence of one ancient Perspective based on the principles rediscovered in the Renaissance. For instance the "Hypothesis of Oxford" (*L'Hypothèse d'Oxford*) of Dominique Raynaud, defends that the invention of the Perspective happened in the thirteenth century, established by the Oxford Philosophers, as Roger Bacon (1214 - 1292) and John Peckham (d. 1292) (RAYNAUD, 1998).

It is possible to distinguish, in medieval and renaissance thought, several conceptions of the Perspective: the *Perspectiva Naturalis*, as "Science of the Vision" (Optics), the *Perspectiva Artificialis* or *Prospectiva Pingendi*, as "Technique of Representation", the *Perspectiva Pratica*, as "Technique of the Measurement" and the *Perspectiva Aedificandi*, related to the architectonical applications (CAMEROTA, 2006, p. 8). In the same way the other terms which had already been analyzed, we can find in some authors distinct meanings to the Perspective.

It was developed in Florence, a transformation of the art concept. The principal people involved in such transformation are Filipo Brunelleschi, Donatello (1386 - 1466), Masaccio (1401 -1428) and Leon Battista Alberti (1404 -1472). Alberti wrote treatises of Painting, Architecture and Sculpture and was the responsible for the theorization of the Perspective, particularly through his work *De Pictura*. In these works he states principles and describes the process of projects to works of art.

According to Giulio Carlo Argan, the humanist thought changed deeply the conception of the space and time:

> The form or the representation according to the reason of the space is the Perspective; the form or the representation according to the reason of the succession of events is the History. Once this succession is not on the things, but it is imposed to the things by the human reason that thinks, there is no difference

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[30] *"Sunt enim quase optima quaedam instrumenta et rudimenta quibus via paratur animo ad plenam philosophicae veritatis notitiam* [...] *Nemo tunc temporis nomine magistri dignus videbatur, qui non harum septem scientiam profiteri posset"* (*Didascalion,* III, 3).

[31] Such tradition has its roots in the biographies of Brunelleschi written by Antonio di Tuccio Manetti (1423 – 1497) and by Giorgio Vasari (1511 – 1574) and confirmed by Erwin Panafsky in his memorable essay *"Die Perspektive als 'symbolische Form'"* (1924).

> between the construction and the representation of the space and the time. The Perspective affords the real space, it means a reality from which is eliminated everything that is casual, irrelevant or contradictory; the History affords the real time, it means a succession of facts in which the occasional, insignificant, irrational is eliminated (ARGAN, 2003, p. 131-132).[32]

The perspectical system of *Quattrocento* is the reduction of the unity in every vision possibilities: the ideal point of localization is the frontal, it means the one which places as opposite, but parallel, the subject and the object. Considering that the Perspective made the rational representation of the natural reality, we can state that it initiated, besides a new artistic phase, a phase in which the reality became understood in mathematical terms.

In the humanistic classification of the disciplines, the Perspective, as science of the vision, was still a philosophical discipline subaltern to the arts of the *Quadrivium.* In the European University of the fifteenth century, the Perspective was generally classified as a case of Practical Geometry. The subaltern position of the Perspective began to be reconsidered from the twelfth century. Domingo Gundisalvo (c. 1100 - 1181), in his work *De Divisione Philosophiae* (c. 1150), considers the Philosophy divided into *scientiae* and the Practical Philosophy besides the Ethics, Politics and Economics, from the Aristotelic tradition, includes the practical disciplines that are related to the Mathematics. In this one, the Perspective is also included (WHITNEY, 1990, p. 133).

Domenico da Chivasso (c. 1350) proposes its inclusion among the arts of the *Quadrivium* too.[33] Michele Savonarola (c. 1385 - 1468), Marsilio Ficino (1433 - 1499), Girolamo Savonarola (1455 - 1498), Luca Pacioli and Leonardo Da Vinci (1452 - 1519) also defended this position. We denominate the debate about the inclusion of the Perspective among the arts of the *Quadrivium* as "Controversy of the Perspective".

4. Luca Pacioli

Frà Luca Pacioli was born in Borgo San Sepolcro[34], *la città di* Piero della Francesca (c. 1416 - 1492), from whom he was disciple and great admirer. Through such great celebrity, he got in touch with the duke Federico da Montefeltro (1422 - 1482) and with his court of Urbino. In this one, he could attend that knowledgeable place that he loved so much: the Library of the Ducal Palace, one of the richest bibliographical collections at that time.

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[32] *"La forma o la rappresentazione secondo ragione dello spazio è la prospettiva; la forma o la rappresentazione secondo ragione del succedersi degli eventi è la storia. Poiché questo ordine non è nelle cose, ma è dato alle cose dalla ragione umana che le pensa, non v'è differenza tra la costruzione e la rappresentazione dello spazio e del tempo. La prospettiva dà il vero spazio, cioè una realtà da cui è eliminato tutto ciò che è casuale o irrilevante o contraddittorio; la storia dà il vero tempo, cioè un succedersi di fatti da cui è eliminato ciò che è occasionale, insignificante, irrazionale".*

[33] *"Est sciendum quaod quinque su[n]t scientiae mathematicae, scilicet arismetrica, geometria, musica, astrologia et perspectiva"* (*Quaestiones super perspectivam*, q. I, f. 44r -v).

[34] Present Sansepolcro, Province of Arezzo, Italy.

Besides of the *Monarca della Pittura*,[35] among those people with whom Pacioli got acquainted with we can mention Leon Battista Alberti, Domenico Bragadino (c. 1430), Albrecht Dürer (1471 - 1528) and Leonardo Da Vinci.[36]

After the publication of his work *Summa di Arithmetica, Geometria, Proportione et Proportionalità* (Venice, 1494), he was invited to belong to the court of Ludovico Sforza (1451 - 1508), duke of Milan. It was exactly in this court that he knew and became a close friend of Leonardo Da Vinci. We can state that they were close friends, because together, they fled from Milan due the French invasion, in 1499. It is quite interesting to notice that beyond the references to Pacioli made by Leonardo in his remaining manuscripts, the note almost puerile made by this great artist: *"Learn the multiplication of roots from Maestro Luca"*.[37]

Pacioli is considered the "Father of Accounting", because he was the first one to publish the "Double Entry Bookkeeping", in his work *Summa.* We can consider this work a kind of encyclopedia which has the meaningful portion of all the mathematical knowledge cultivated until its publication day. Maybe Pacioli had not contributed directly with the great mathematical discoveries, as some authors insist on saying. However, it is original his work of compilation and his works are indispensable references to the study of the History of the Mathematics in the Renaissance. Besides, we consider that Pacioli had an important role in the development of the elementary Algebra, since his work was a stimulus for Girolamo Cardano (1501 - 1576), Niccolò Tartaglia (c. 1500 - 1507), among others, to make their investigations in order to discover general solutions to polynomial equations of degree 3 and higher.

4.1 Works of Luca Pacioli
Following the list of the known works of Luca Pacioli.

4.1.1 Manuscripts
1. Treatise of Algebra. Ms. dedicate to the sons of Antonio Rompiasi. Venice, 1470. Current whereabouts unknown.
2. *Tractatus ad discipulos Perusinos.* Perugia, 1476. Ms. Vat. Lat. 3129. *Biblioteca Vaticana*.
3. Treatise of Arithmetic. Zara, 1481. Current whereabouts unknown.
4. *De Divina Proportione*. Milan, Dec. 9th 1948. Ms. *Bibliothèque Publique et Universitaire de Genève*.
5. *De Divina Proportione*. Milan, Dec. 14th 1948. Ms. 170 sup.. *Biblioteca Ambrosiana* of Milan.
6. *De Divina Proportione*. Ms. dedicated to Pier Soderini, c. 1500. Current whereabouts unknown.

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[35] "[...] *el monarcha ali t[em]pi nostri d[e]la pictura maestro Pietro di Frãceschi n[ost]ro cõterraneo [...]"* (PACIOLI, *Summa*, f. 2r).

[36] For a study about Luca Pacioli an his time v. TAYLOR, 1980.

[37] *"impara la multiplicatione delle radice da maesstro Luca"* (*Codex Atlanticus* 118a; 366a acc. RICHTER, 1970, v. II, p. 433; 120r-d acc. PEDRETTI, 1957, p. 44).

7. *De Viribus Quantitatis.* Codex n. 250. *Biblioteca Universitaria di Bologna*. 309 folios.
8. *De ludo scachorum* or *Schifanoia*. Ms. dedicated to Marquis of Mantua, Francesco Gonzaga, and his wife, Isabella d'Este.[38]

4.1.2 Printed Works

1. *Summa de Arithmetica, Geometria, Proportioni et Proportion*alità. Venice: Paganinus de Paganini, Nov. 20th 1494.
2. *De Divina Proportione*. Venice: Paganinus de Paganini, Jun. 1st 1509.
3. *Summa de Arithmetica, Geometria, Proportioni et Proportionalità*. Toscolano: Paganinus de Paganini, Nov. 10th – Dec. 20th 1523.
4. *Euclidis Megarensis Philosophi acuratissimi... a Campano... tralata.* Venice: Paganinus de Paganini, Jun. 11th 1509.
5. Translation to the Italian of the *Euclid's Elements*. [1509]. No copy known.

4.2 Mathematics and Perspective according Luca Pacioli

In his work *De Divina Proportione*,[39] published in 1509, Pacioli explains that the term $maqhmatikov''$ derives from the Greek and that, in his idiom, it is the same as "disciplinable" (*"discipinabile"*). He considers that the sciences and the mathematical disciplines (*"scientie e discipline"*) are, to his proposal, Arithmetic, Geometry, Astrology (or Astronomy), Music, Perspective, Architecture, Cosmography and any other which depends on this ones (PACIOLI, 1498, *De Divina Proportione*, III, f. 9r-v). As we can see, this list is much more extensive than the *Quadrivium*, considering the subaltern disciplines as well. For him, the mathematical science, are in the first degree of certainty.[40] Without his knowledge is impossible to understand well any other science, once, everything that is distributed in the inferior and superior universe, converts necessarily into the number, weight and measure.[41]

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[38] Manuscript about the chess game, discovered in the library of *Fondazione Palazzo Corini Cronberg di Gorizia*, with about 20 thousands volumes. The bibliophile and historian Duillio Contin, identified the work among the manuscripts of the Earl Guglielmo Coronini (1905 – 1990). Casually, Serenella Ferrari Benedetti, coordinator of the Fundation, showed him a manuscript of an unknown author. The discovering was confirmed by the exam of the paleographer Attilio Bartoli Langeli and Enzo Mattesini, professor of Italian Linguistic in the *Università di Perugia*. The manuscript *De Ludo scachorum* was bought by the *Conte* Guglielmo Coronini, in 1963, in a bookstore which belonged to the poet and bibliophile Giuseppe Malattia della Vallata.

[39] For a general vision about the mathematical contents of the *De Divina Proportione* v. BERTATO, 2004.

[40] *"Concio sia che ditte mathematici sieno fondamento e scala de peruenire ala notitia de ciascuna altra scientia : per esser loro nel primo grado dela certezza affermandolo el philosopho cosi dicendo mathematice enim scientie sunt in primo gradu certitudinis & naturales sequuntur eas. Sonno como e dicto le scientie e mathematici discipline nel primo grado dela certezza e loro sequitano tutte le naturali : e senza lor notitia fia impossibile alchunaltra bene intendere"* (*Divina Proportione*, II, f. 5r). Pacioli attributes the authorship of this statement to Aristotle, even referring to Averroes before (cf. *De Divina Porportione*, I, f. 1v). Actually it is a translation from a passage of the commentary of Averroes on *Metaphysica*, II, 3, 995a, 15: *"Demonstrationes .n. Mathematicę sūt in primo ordine certitudinis: & demōstrationes Naturales consequūtur eas ĩ hoc"* (ARISTOTLE, 1562, f. 35v).

[41] *"Omnia consistunt in numero, pondere et mensura"*. Variation from the Latin text of the Book of Wisdom 11, 21.

As in the Chapter II of the *De Divina Proportione* as in the Epistle to Guidobaldo of Montefeltro,[42] which is part of the *Summa*, Pacioli states that the mathematical disciplines are applied in the following areas: 1) Astrology; 2) Architecture; 3) Perspective; 4) Sculpture; 5) Music; 6) Cosmography; 7) Commerce; 8) Military Arts; 9) Grammar; 10) Rhetoric; 11) Poetry; 12) Dialectic; 13) Philosophy; 14) Medicine; 15) Civil and Canonic Law and 16) Theology (cf. PACIOLI, 1494, f. 2r; PACIOLI, 1498, f. 4r-9r).[43] It is clear the worry with the applicability of the Mathematics and the superiority of this one related to the others, once, according to him, only the sciences and mathematical disciplines can be called certainties (cf. PACIOLI, 1498, I, f. 3v), being the others just opinions.

Pacioli divides the mathematical sciences and disciplines in Practical and Speculative. The Algebra, denominated by him *Pratica Speculativa*, is a special case of Practical of Arithmetic and Geometry. The Great Art is the Algebra and the Minor Art is the *Pratica Negotiaria* (Practical Commercial).[44]

In his work *Summa*, in the *Distinctio Octava* dedicated to Geometrical issues, Pacioli deals with an issue pertinent to the Perspective, where he states that this is one subaltern discipline to the Geometry and the Arithmetic:

> This is an issue of the Perspective, but as this science is subaltern to the Geometry and the Arithmetic, we will solve it (PACIOLI, 1494, *Summa*, *Distinctio octava*, Cap. II, f. 65r).[45]

It is in the *De Divina Proportione* that Luca Pacioli shows explicitly his mystics conceptions about the Golden Ratio or the Divine Proportion. It is also in this work that he positioned about the "Controversy of the Perspective".

4.3 The *Quadrivium* and the "Controversy of the Perspective"

As it has already said, Pacioli defends the elevation of the Perspective at the same *status* of the arts of the *Quadrivium*. Among the arguments presented by him in defense of the Perspective, the exaltation of the sight deserves special attention:

> It is among our senses, the wise men conclude, that the sight is the noblest one. That is why vulgarly said, with reason, that the eye is the first door from which the intellect understands and likes (PACIOLI, *De Divina Proportione*, f. 4r).[46]

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[42] *Alo Illu^mo. Principe Gui.Baldo. Duca de Urbino. Epistola.*

[43] The same structure of argumentation is found in the speeches of Niccolò Tartaglia (*Lettione de Nicolo Tartalea Brisciano, sopra tutta la opera di Evclide Megarense, acvtissimo mathematico*) which is seen in the beginning of his translation of the *Elements*, besides the reference to the friar.

[44] *"Non mi pare ormai piu douer diferire la p[ar]te maxime necessaria ala pratica de arithmetica e anche de geometria detta dal vulgo cõmunemente. Arte magiore ouer. La regola de la cosa ouer. Algebra. E almucabala secõdo noi detta pratica speculativa. Per che in lei piu alte cose che in larte minore ouer pratica negotiaria si cõtiene"* (PACIOLI, *Summa*, f. 111v).

[45] *"Sapi che questa domanda è de perspectiva, ma perché questa scientia è subaltternata a geometria e aritmetrica si la solveremo".*

[46] *"E deli nostri sensi per li sauii el uedere piu nobile se conclude. Onde non immeritamente anchor de uulgari fia detto lochio esser la prima porta per la qual lo intellecto intende e gusta".*

Pacioli calls the sight as "the first door from which the intellect understands and likes". Such statement refers to a passage of the work of Feo Belcari (1410 - 1484), entitled *Rappresentazione di Abramo e Isacco*:

> *Lo occhio si dice che è la prima porta*
> *Per la quale lo intellecto intende e gusta.*
> *La secunda è lo audire con voce scolta*
> *Che fa la nostra mente essere robusta* (BELCARI, 1833, p. 3).

The Hellenic tradition took some authors to an exaltation of the sight as primate among the senses. Heraclitus affirms, *"the eyes are witness more accurate than the ears"*.[47] In Timaeus of Plato we find: *"The sight, in my opinion, is the cause of the great enjoyment for us"*.[48] Aristotle says, *"The sight is the most highly developed sense"*[49] and there are several others references about this theme (cf. ARISTOTLE, *De Generatione et Corruptione*, II, 2, 329b, 14; *Metaphysica*, I, 1, 980a, 25 e *Ethica Nicomachea*, X, 5, 1176a, 1). For Pacioli, the wonder is possible through the vision, first of the senses, and as *"by the wonder began the philosophizing"* (cf. ARISTOTLE, *Metaphysica*, I, 2, 982b, 12) from there comes the statement *"from the sight the knowledge comes"*[50], supported by the authority of Aristotle.

Similar argumentation is shown by Leonardo Da Vinci, in his *"Paragone"*[51], where he states that the eye, *"which is said window of the soul"*, is the principal course where the infinite masterpieces of the nature can be considered.[52] For Pacioli and Leonardo, the sight is the beginning of the knowledge, because *"there is nothing in the intellect that was not first in the senses"*,[53] and the first of the senses is the sight. For Leonardo, the vision embraces the beauty of the whole world; the eye is the "Prince of the Mathematics". For both, there was not reason to consider the Music as mathematical discipline and ignore the Perspective.

The Chapter I of the *De Divina Proportione* shows to the reader one description of the court ambient of Milan, in the time of Ludovico Sforza.[54] In this Chapter, Pacioli reminds the *"scientifico duello"* (scientific disputation), a debate occurred in February 9th 1948, with the participation of illustrious persons of the time, among those, Leonardo Da Vinci. Pacioli

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[47] *"ovfqalmoi. ga.r tw/n w;twn avkribe,steroi ma,rturej"* (PLUTARCH, *Adversus Colotem*, 1118C).

[48] *"o;yij dh. kata to.n evmon lo,gon aivti,a th/j megi,sthj wvfeliaj ge,gonen h`mi/n"* (PLATO, *Timaeus*, 47a).

[49] *"As sight is the most highly developed sense..."* (ARISTOTLE, *De Anima*, III, 3, 429a, 4, trans. J. A. Smith).

[50] *"dal uedere hauesse initio el sapere"* (PACIOLI, 1498, f. IIIIr).

[51] It is called *Paragone* the sequence of polemic disputes among the Painting and some of the others arts which is found in editions of *Trattato della Pintura* of Leonardo.

[52] *"L'occhio che si dice finestra dell'anima, e la principal uia donde il comune senso po piú coppiosa et magnificamente considerare le infinite opere de natura et l'orecchio è il secondo, il quale si fa nobbile per le cose raconte, le quali ha ueduto l'occhio, se voi istoriograffi, o poeti, o altri matematici, non hauestiue con l'occhio uisto le cose, male le potresti uoi rifferire per le scritture."* (LEONARDO DA VINCI, *Codex Urbinas Latinus* 1270, f. 8r, *Trattato*, 15).

[53] *"Nihil est in intellectu quin prius fuerit in sensu: cioe che niuna fia nel intellecto che quella prima non se sia per alchun modo al senso offerta"*(PACIOLI, 1498, f. IIIr. cf. ST. THOMAS AQUINAS, De Veritate, q.2, a.3, arg. 19).

[54] It is a dedicatory letter to the duke titled *"Excellentissimo Principi Ludovico Mariae Sforciae Anglo Mediolanensium Duci, Pacis et Belli Ornamento, Fratris Lucae Pacioli ex Burgo Sancti Sepulchri Ordinis Minorum, Sacrae Theologiae Professoris, De Divina Proportione Epistola"*.

give him great praises and affirms that this one had already concluded "the worthy book of Painting and human movements".[55]

It is important to observe that the "*scientifico duello*" of Pacioli and the "*Paragone*" of Leonardo seem to be complementary. It is noted several similarities and we can suppose that the court of Milan had been place of a serie of debates about what would be more important the science or the art. Unfortunately, besides reasonable details of the initial chapters of the *De Divina Proportione,* we do not know the existence of some text where Pacioli shows extensive and elaborated arguments about the "Controversy of the Perspective", as the ones done by Leonardo about the "Dispute of the Arts".

Monica Azzolini, in two recent works (AZZOLINI 2004 and 2005), makes an interesting analysis of the dynamic of the scientific patronage in the Renaissance and the social and economical changes of the ones involved, from the "*scientifico duello*" and the "*Paragone*". According to her, "*by participating in the duel, Leonardo and Pacioli challenged the traditional hierarchy of disciplines and, at the same time, the social, economical and intellectual status that indissolubly came with it*" (AZZOLINI, 2004, p.128). Besides the relevance of her approaching, such discussions escape from the aim of our article, so we do recommend the reading of her articles for an ampler understanding of the context of the "Controversy of the Perspective".

"*The ones who are not mathematicians do not read me, neither my principles*".[56] Such statement, similar as the written in the portico of the Academy of Plato, evidences the role of the Mathematics in the work of Leonardo Da Vinci. For him, "*no human investigation can be called true science if it does not go through mathematical demonstration*".[57] For Leonardo, the Painting is the true "*scientia*" and is based on the mathematical foundations.

We can observe a change in the thought of Leonardo about the Perspective. Sometimes the Perspective is the "daughter of the Painting"; other times it is its "rein and rudder". In

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[55] "*hauẽdo gia cõ tutta diligẽtia al degno libro de pictura e mouimenti humani posto fine*" (*Divina Proportione*, f.2r). This statement is frequently cited by the historians interested in Leonardo Da Vinci and his masterpieces. The writings of Leonardo are found in several notebooks and, in 1883, they were joined by Jean Paul Richter and published with the title of *The Literary Works of Leonardo Da Vinci*. Richter states, based on the assertion of Pacioli, that the researches of Leonardo about the proportions and movements of the human figure should have already been, in most of the cases, finished and written in 1498 (RICHTER, 1883, Cap. VII). Vasari, in his work *Le vite*, says that some of the written of Leonardo, which referred to the painting and the method of the drawing and use of the colors, were with a painter of Milan and such painter wished to publish them. ("*Come anche sono nelle mani di... pittor milanese alcuni scritti di Lionardo, pur di caratteri scritti con la mancina a rovescio, che trattano della pittura e de' modi del disegno e colorire. Costui non è molto, che venne a Fiorenza a vedermi, desiderando stampar questa opera, e la condusse a Roma per dargli esito, né so poi che di ciò sia seguito*". This passage is not found in VASARI, 1550, only in VASARI, 1568). This writtings could have the notes of Leonardo which got nowadays among those which are the transcriptions that are in *Biblioteca Apostolica Vaticana* (*Codex Urbinas Latinus 1270*) and are known as *Tratatto della Pintura*. In the third part of this copy, which deals with the movement and proportions of the human being, we can find the following words: "*e il resto si dira nella universale misura del huomo*". This "universal measurement of the man" can be the "worthy book" mentioned by Pacioli.

[56] "*Nõ mi legga chi non è matematicho nelli mia prîcipi*" (RICHTER, 1970, p. 11).

[57] "*Nissuna humana inuestigatione si po dimandare uera scientia se essa nõ passa per le Matematiche dimostrationi.*" (LEONARDO DA VINCI, *Codex Urbinas Latinus* 1270, f. 1v, *Trattato*, 1).

another place he states that *"the Painting is based on the Perspective, it not but only a tiny knowledge of the eye"*.[58]

Final Considerations

This work is a result of researches made to the development of the PhD thesis – "*De Divina Proportione de Frà Luca Pacioli* (Translation and Commentaries)" – from the first author, under the guidance of the second author, in conclusion, at the State University of Campinas (UNICAMP). In this thesis, we will examine thoroughly the study and the work of Pacioli, mainly the *De Divina Proportione.* From this work, up today, there are only translations in French, Germany or Spanish. As a result, we intend to make able one translation commented into Portuguese and, further into English, from the referred work.

Besides many authors, debates and standings along the History, about the classification of the Mathematics and the science, the *Quadrivium* in its several acceptations and interpretations, performs a fundamental paper in this discussion and in the contemporary thought.

We tried to focus the Pythagorean origin of the *Quadrivium,* its assimilation in the Platonic and neo-Platonic thought, to contextualize the discussion which becomes stronger in the Renaissance, mainly in the works of Luca Pacioli and Leonardo Da Vinci.

We believe that the theorization of the Perspective has large repercussion in the scientific thought, allowing the development of the Projective Geometry and showing a new conception of space, necessary to the development of the modern science.

The position of Pacioli, Leonardo Da Vinci and others, concerned to the Mathematics and the Perspective can be considered as precursor of the summarized conception attributed to Galileo Galilei: "*La matematica è l'alfabeto nel quale Dio ha scritto l'universo*".

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[58] *"La pittura effondata sula che[lle] prospectiva. non è altro che sapere bene figurare lo vfitio dell'ochio"* (LEONARDO DA VINCI, A. 3a, RICHTER, 1970, v. I, p. 29).

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