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PEIRCE'S MATHEMATICAL WRITINGS: AN ESSAY ON *PRIMARY ARITHMETIC* BOOKS AS IT RELATES TO MATHEMATICS EDUCATION [1]

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Resumo

Charles Sanders Peirce é um dos mais importantes filósofos americanos. Embora a literatura educacional, com raras exceções, negligencie esse aspecto, sabe-se que o Pragmatismo peirceano, por exemplo, é o ponto de partida de James e Dewey tendo, portanto, clara influência nos tratados educacionais. Os escritos matemáticos de Peirce, tanto quanto sua filosofia, são, segundo nosso ponto de vista, importantes sob dois aspectos: (1) como fonte de pesquisa em História da Matemática e (2) como recurso para compreendermos as concepções peirceanas acerca da Matemática e os processos de ensino e de aprendizagem dessa disciplina, uma questão claramente ligada à História da Educação Matemática. Neste artigo focamos especificamente sua produçào matemática, estudando os manuscritos da **Primary Arithmetic** e textos correlatos, numa tentativa de fundamentar nossas afirmações. Originalmente escrito em inglês, este artigo foi produzido durante nosso estágio de pós-doutoramento junto à Indiana University Purdue University, Indianapolis, Indiana, Estados Unidos

Abstract

Charles Sanders Peirce is one of the most important and influential American philosophers. His pragmatism was a special starting point for James and Dewey. But there is an almost embarassing lack of research focusing on Peircean theories in the educational literature. We intend to illustrate this importance from two distinct - but complementary - perspectives. We claim - and this is the first perspective - that pragmatism is a fruitful philosophical doctrine to be implemented as a theoretical foundation for research concerned with cognition and other elements related to the teaching and learning of mathematics. The second perspective – the perspective which is specifically discussed in this paper – indicates the importance of Peirce's mathematical works in two aspects: (1) as a source of research on the history of mathematics, and (2) as a source to understand Peirce's conception about mathematics teaching and

[1] This paper was originally written in English during the period the author spent as a post-doctoral fellow at Indiana University Purdue University, Indianapolis, Indiana, USA.

learning processes, an issue related to the history of mathematics education. In order to support our second perspective, Peirce's **Primary Arithmetic** and correlated manuscripts are analyzed.

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1. Peirce's mathematical manuscripts: intentions and references

As we understand it, there are two aspects in Charles Sander Peirce’s academical production which are powerful tools to comprehend the process of teaching and learning Mathematics: on the one hand, Peircean Pragmatism and theory of signs in which a new philosophy of Mathematics Education can be rooted[1] and, on the other hand, Peirce’s mathematical writings. Although Peircean pragmatism and theory of signs can speak for themselves, in another paper we highlight his conception about truth, cognition, reasoning, science and reality among others, which – according to our point of view - allow new approaches to old themes, opening up perspectives. But there is another aspect we can consider when looking for possibilities for using Peirce's theory in mathematics education. This other aspect of Peircean writings we realize as important to mathematics education is the understanding of Peirce's mathematical production, so to speak. A presentation of one of these manuscripts - the primary arithmetic and correlated manuscripts - is made below. But first some remarks on the sources for references on Peirce must be made.

The eight volumes of the ***Collected Papers of Charles Sanders Peirce***, published by Harvard University Press, have been one of the most important sources for references on Peirce. But since 1958 - the year Arthur Burks edited the last volume - some other editions became public, complementing the Collected Papers. Surely it demonstrates the increasing academic interest on Peircean works, but at the same time it represents a problem for making unambiguous reference citations on the original manuscripts. Nowadays we need to recognize the efforts of Indiana University's Peirce Edition Project in publishing the complete writings of Peirce in a chronological edition (6 volumes available, 30 projected) and other initiatives which will naturally appear in this paper. In order to make references clear and facilitate reading, we use a special notation for all Peirce's original quotations, noting down in capital letters the editions used followed by the page numbers. We'll use CPn to indicate volume n of "Collected Papers", CEn for the volume n of "Chronological Edition", EPn for the volume n of "The Essential Peirce", NE for "The New Elements of Mathematics", EW for "Essential Writings", SE for "Selected Writings", PW for "Philosophical Writings" and so on. Exceptions in notation will be explicitly mentioned and the complete list and general data are available in the final bibliographic references.

2. Mathematical manuscripts: context and history

Charles Sanders Peirce was born in 1839 in Cambridge, Massachusetts. E. T. Bell, G. Birkhoff and D. Struik are among the researchers who agree on the contribution Benjamin Peirce, his father, brought to American mathematics. According to Bell, mathematics in the United States was in a state of sterility during the first three hundred years of its existence.

An honorable exception must be credited to Benjamin's linear associative algebra, later recognized:

> "*Sylvester's enthusiasm for algebra during his professorhip at the Johns Hopkins University in 1877-1883 was without doubt the first significant influence the United States had experienced in its attempt to lift itself out of the mathematical barbarism it appears to have enjoyed prior to 1878. Elementary instruction was not good enough, perhaps better than it is today, research on the European level, with one or two conspicuous exceptions, was non existent. /.../ Benjamin Peirce made only a negligible impression on his American contemporaries in algebra, and his work was not appreciated by their immediate successors until it had received the nod of European condescension.*" (NE1, p.xiv)

In Peirce's time, there was a need for applied mathematics, and the mathematically talented were easily attracted to research centres. Charles S. Peirce entered Harvard as a student in 1855, the same institution in which his father had been one of the most distinguished professors in Mathematics and Astronomy. In 1859, he became a member of The Coast and Geodetic Survey, the first scientific institution to be created by the government of the United States. Peirce's trajectory in the scientific community was agile: he was also a member of the National Academy of Sciences, and taught logic at Johns Hopkins University from 1879 to 1884. He died in 1914 in Milford, Pennsylvania.

Although almost unknown by mathematics educators and researchers, Peirce's mathematical works are highly regarded creative texts. According to specialists in his manuscripts, Peirce is an inspired textbook writer, who had challenged the French influence of his time. In those days, Legendre's ***Élements de Géométrie*** and ***Traité de Trigonométrie*** were the pattern to be followed. In spite of this model, Peirce's textbooks for elementary and advanced mathematics have a clear and logical discussion of ideas and concepts, using - and sometimes creating - a powerful and inventive symbology linked to a careful nomenclature ("reflecting his work as a linguist and a contributor to dictionaries" as quoted in NE1, p.xxvi). In his elementary arithmetics, Charles Peirce even anticipates the curricular revision suggested by the commission responsible for recommending some changes in mathematics education in 1908:

> "*Textbooks written and used in elementary mathematics in America up to the time of C. S. Peirce's involvement in the problem reflected little the revolutionary mathematical thought of the mid-nineteenth century. However, by the end of the century the need of a review of mathematical curriculum and instruction throughout the world became apparent, and steps were taken at the International Congress of Mathematicians in Rome in 1908 to implement just*

> *that. A commission with Felix Klein at the head was appointed to make recommendations for the necessary changes. Peirce antecipated such revision in his own textbook writing /.../*" (NE1, p. xxvi)

Ingenious, Charles Peirce's mathematical work spans a wide range in the mathematics landscape, from the basics - developed under different perspectives - to the most sophisticated from a mathematical point of view. Peirce's arithmetic textbook is a brilliant example of the former while his Topology could be an example of the latter:

> "*/.../ his deep appreciation of topological structure at a time when nothing was being written to introduce the basic topological ideas on the lower school level, and little on a higher level, his fascination with non-Euclidean notions that is reflected in the appearance of the Moebius strip in his geometry even though Klein himself had advised against the introduction of non-Euclidean concepts on so low a level in his Evanston lectures, all tend to make of C. S. Peirce a mathematical prophet, as well as a superb mid-twentieth century teacher*[2].
> *Yet topology in the 1890s, like non-Euclidean geometry, had not reached textbook recognition; unlike the non-Euclidean materials, little of it was to be found in widely circulated research papers.*" (NE1, p. xxvi-xxvii)

According to Grattan-Guiness (1997) in his study on the interactions between mathematics and logics from the French Revolution to the First World War, Peirce is considered one of the most important logicians among those followers of Condillac's ***Logique***. Such relations between mathematics and logics are quite confusing in the cited period for the scene has many "actors" in supporting roles. We must also consider the specific philosophical and religious convictions involved here. By the way, Grattan-Guiness analyses the influence of Condillac's *logique* in France during the French revolution and its posterior arrival in England, where Boyle and De Morgan establish the foundation of Algebraic Logics. This line of development has its continuation in Peirce and Schröeder, both using boolean algebra with interpretations in a propositional calculus and a theory of collections. Peirce, developing his theory of relations and a general theory of signs, still revived the connections between logics and semiotics forgotten since Condillac's *logique*.

Being concerned with many and distinct aspects of scientific and philosophical knowledge, Peirce had an amazingly large written production scattered among papers, letters and notes. Among the important projects related to his work, seeking a systematic way to present the production of one of the most important American philosophers, we can cite the books edited by Charles Hartshorne and Paul Weiss[3] (the ***Collected Papers of Charles Sanders Peirce***, Harvard University Press); the complete manuscripts, chronologically ordered, developed by the Peirce Edition Project (the ***Chronological Edition***, Indiana

University Press) and the presentation of Peirce's mathematical works edited by Carolyn Eisele (***The New Elements of Mathematics by Charles S. Peirce***, Humanities Press). In the four volumes of the ***New Elements***, we can find Peirce's arithmetics, algebra and geometry and some of his remarks on philosophy of mathematics. This edition, published in 1976, continued to be an obligatory source for understanding Peirce's thinking about mathematics and his concerns about the process of teaching and learning of mathematics. Actually, according to our point of view, his works on arithmetic are among his most interesting mathematical writings because of the way in which they show the approximation of the man of science with basic educational concerns. However, these writings have a chaotic history of comings and goings from one editor to another, from one collaborator to another, and due to unsolved financial issues, excessive spending and lack of time, a complete final version of Peirce's arithmetic never was published. In spite of this, the manuscripts were organized by Eisele in her 1976 edition. Eisele herself gives us a broad understanding of the composition of those texts:

> "Peirce had in mind at that time a "Primary Arithmetic" consisting of the Elementary Arithmetic as given in MS. 189[4,5] (***Lydia Peirce's Primary Arithmetic***) and MS. 181 (***Primary Arithmetic*** - MS. 182 is a draft of 181 with Suggestions to Teachers); a ***Vulgar Arithmetic***, as developed in MS. 177 (***The Practice of Vulgar Arithmetic***) for students and in MS. 178 (C.S.Peirce's Vulgar Arithmetic: its chief features) for teachers; a Practical Arithmetic as given in MSS. 167 and 168. In an "Advanced Arithmetic", he probably intended to encompass number theory as given, for example, in Familiar Letters about the Art of Reasoning (MS. 186) and in ***Amazing Mazes***; and ***Secundals***, the binary number system so popular today." (emphasis added).(NE1, p. xxxv)

This paper intends to present some remarks on Peirce's ***Elementary Arithmetic***, consisting of manuscripts 189[6], 181, 182 and part of MS 179 (Peirce's ***Primary Arithmetic upon the Psychological Method***).

3. Peirce's Primary Arithmetic

The arithmetic in Peirce's manuscripts seems to be a skeleton of a textbook - sometimes more complete, sometimes with gaps - which was supposed to be actually adopted in elementary schools. We detect some explicit directions to teachers in terms of methodology. In manuscript 179, for instance, Peirce establishes the scientific posture as one of the most essential characteristics teachers should have:

> "*It had already been recognized that numerals are not learned by children in the same involuntary way in which they seem to learn the other parts of speech. They have to be taught number; and it is almost indispensable to their future facility with arithmetic that they could be taught in a scientific manner, so*

> *as not to burden their minds with fantastic notions. /.../ If the teacher cannot prevent the formation of associations so unfavorable to arithmetical facility, as in many cases he certainly cannot, he can at least do something to give them the least disadvantegeous peculiarities. To this end, it is desirable that children should receive their first lessons in number from an instructor conversant with the dangers of these phantasms*[7]*.*"

In an uncertain time, Peirce creates a story and situates its characters - a little girl called Barbara (in a clear reference to the classic syllogism) and her grandmother Lydia[8] - talking about numbers. In the beginning, the text has the characteristic language of children's tales, a resource the author gradually abandons as the text progresses:

> "*Once upon a time, many, many, many long years ago, when the world was young, there was a little girl /.../ who lived in the midst of a great wood; nothing but trees, trees, trees, in every direction for further than I could tell you until you have learned arithmetic /.../*".

Peirce is very careful with language issues and this can be easily detected in the rhythmic cadence of some phrases:

> "*One of the things we have to do very often is to find out how many things of the same kind there are in some box or bags, or basket or barrel, or bank or basin or bucket, or bureau, or bottle, or bowl, or bunker, or bird's nest, or buffet, or boiler, or barrow, or barn-bay, or book, or be it what it may, or to find out how many times anything happens, or any other kind of how many.*"

Counting and basic operations are connected, and while pointing out the importance of some hands-on materials - like cards, drawings, tiles and beans - to support the teaching-learning situation, Peirce takes the opportunity to teach lessons of another nature:

> "*If we don't want to make people sorry but want to make them glad, we must begin by finding out what the right way is /.../ That makes three things: 1st to find out the right way; 2nd to learn the right way and third to do the right way.*"

The concept of counting is introduced using the well known resource of children's games by relating childhood counting-rhymes[9] to a kind of biunivocal relation:

> *"Have you never picked the petals from a daisy and said 'Big-house, little house, pigsty, barn; big-house, little house,' and so forth. Then the last one called is supposed to be your future home. That's like counting."*

Reminiscing children's rhymes, telling facts of American history and conjecturing about the past[10], Peirce intends to keep tradition alive[11].

The widely used mnemonic resource is not a mere technique to facilitate calculation, but is justified by the very fact that it gives to the child a close and concrete reference. The same resource can be seen when Lydia teaches Barbara and Benjie how to count using their fingers[12], but after this, using some examples, the meaning becomes clear. Multiplication, division, average (arithmetic mean) and rule of three are issues discussed in the first arithmetics book. As an example, we will take here Peirce's remarks on multiplication of integers.

An initial element to be noted is Peirce's definition of integers from 0 to 18. These numbers are shown in a list from which we can figure out how to operate to get the quarter square of n:multiply the integer part of half n by what is missing from that first factor to complete n[13]. In this way, Peirce intends to be constructing an agile mnemonic process for multiplication of integers smaller than 10[14], which we could call "short multiplications". More ingenious than practical or easy, the further algorithm presented is the one to get the product of multiplying n by m: it is given taking the difference between the quarter square of m+n and |m-n|. The next step in this line of presentations of algorithms is that related to "long multiplications"[15].

A kind of conflict resolution problem related to American history which requires the result of 365 multiplied by 127 is the starting point to discuss "long multiplications". The technical nature of the algorithm - which is quite close to that which is used in schools nowadays - consists of transforming a "long" multiplication into as many "smaller" ones as necessary and, finally, adding the final results of these partial products[16]. This first approach to long multiplication is developed in such a way that it is not necessary to use the processes of "carrying" numbers.

As a result of this process, however, the need arises to understand the position of the partial results on the graph, and afterwards, to understand the need to sum these values. This explanation, also in the form of examples, emerges in a dialogue between Benjie and Lydia[17]. In the sequence, the algorithm is presented in its most agile form: it develops like the previous algorithm, placing "units under units, tens under tens, and hundreds under hundreds", but now making use of the process of carrying numbers[18], formally recording the carried numbers in their proper positions. In a next - and final - step of the lesson on the multiplication of whole numbers, these records will be disregarded[19].

Additional unlocking-problems are proposed and the way shown and, thus, the theory is developed. Peirce's advice to teachers is clear, and at many points, quite similar to modern thinking with respect to mathematical literacy. The importance of complementary didactic materials (such as colorful chips, bags, abacuses, diagrams, and cards) is emphasized, as well as elements familiar to the students' context, in order to bring them closer to the idea of number (which, according to Peirce, many teachers do not have themselves) in a way that is natural, clear, simple, and useful, indicating that "It must not be supposed that so long as children learn arithmetic, it makes no difference how they learn it", which can be read as valuing the process over the result.

In one part of the original text (***Peirce's Primary Arithmetic: upon the psychological method***), the author declares that, although he learned his elementary arithmetic lessons with his parents, the contact with modern psychology made him see certain concepts and positions with new eyes, which helped him to perfect his lessons.

In manuscript 179 he presents Miss Sessions and her students, in whose lessons he continues to develop the same themes but from a different perspective, using new problems. Some private notes of Peirce's are incorporated in the manuscript and show that the author analyzed various didactical texts of arithmetics commonly used in the American educational system at the time, pointing out their omissions and inconsistencies in an effort to caution himself and others about them[20].

The need that Peirce saw to take advantage of arithmetic lessons to, at the same time, comment on cultural and historical aspects is notable. The presence of elements that, like current educational guidelines, suggest an almost obligatory link between mathematics and the day-to-day lives of students, should also be emphasized.

The study of the arithmetic mean, in manuscript 189 (second version) begins with a comparison of the Julian and Gregorian calendars; the part of manuscript 179 related to Primary Arithmetic begins with Ms. Sessions going shopping with her students, from which the lessons about linear measures and addition and subtraction problems involving money evolve. The elements of mathematics must be taken advantage of to teach other lessons linking science with that which, implicitly or explicitly, motivate and strengthen notions of ethics in the students. In addition, the use of support materials is constant[21], taking care that the material not be more attractive than the concept to be taught. Examples of such materials include the abacus, which serves not only for rapid calculations, but also as a trigger for linguistic considerations related to the name of things[22]. From this activity follows counting "by twos" and "by threes", etc., from which emerges the list of multiples that forms the basis for the multiplication table.

> "*There are no better diagrammatic presentation of a number than a row of dots, all alike. For this reason, the usual abacus with round beads on wires is to be commended. The beads should be spherical, or somewhat flattened, in the direction of the length of the wire, so that their form may attract as little attention as possible. They should all be of one color, so as to avoid insignificant associations of colors with number. We must not fail, in teaching numbers, to show the child, at once, how numbers can serve his immediate wishes. The school-room clock should strike; and he must count the strokes to know when he will be free. He should count all stairs he goes up. In school recess playthings should be counted out to him; and the same number required of*

> *him. This is to teach the ethical side of arithmetic. /.../ In counting, the child should begin by arranging his pack of cards in regular order, and then laying down a card*[23] *upon each object of the collection to be counted. In this way, he will count articles of furniture, flower-pots, plates, books, etc.*" (MS. 179)

We cannot know from the ordering of his manuscripts (and perhaps his own composition, since they are drafts that were never put in their final form) what he had in mind to be the exact sequence and desired pre-requisites. There are few clues in this respect: manuscript 180, reproduced here in one of the footnotes, suggests a work plan that other manuscripts attest was not followed, at least not rigorously. It should be noted, nonetheless, that in spite of being material destined for the schools, to which each student would have access[24], it serves as a guide for teachers, who can take it as a reference of precise guidelines for organizing their actions.

Using strong mnemonic resources, such as exhaustive repetition and rhythmic lists of numbers, his attempt to teach arithmetics is based on the strategies of reinforcing counting (with or without the support of complementary material), enunciating the sequence of whole numbers, and counting in two's and three's, etc., and in the intent to form, in the student, a solid base for agile mental calculations. Thus the intention, as Peirce himself affirms in one of his letters, is not the constitution of arithmetics in the broad sense, but of arithmetics thought of as the art of using the Arabic numerals and managing the principles of counting:

> "*But now as to Arithmetic, which, properly speaking, has nothing to do with arithmetic, the mathematics of number, but solely with the art which Chaucer and others of his time called augrim, the art of using the so-called 'Arabic' numeral figures, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. It is a great pity that the word 'augrim' has become obsolete, without leaving any synonym whatever. The nearest is 'logistic', which means the art of computation generally, but more especially with the Greek system of numeral notation.*"

Also, in his brief considerations of the teaching of numeration (MS. 179), Peirce affirms that work with counting is the best - or only - possible way to approach the concept of number: "*The way to teach a child what number means is to teach him to count. It is by studying the counting process that the philosopher must learn what a essence of number is*".

In joining numerals together with childhood games, one could suppose that the names of these numerals are thought of as being mere words; this is followed by encouragement to recite the rhymes and sequences with the intention of solidifying and learning[25].

When the utilization of the abacus is suggested, the concern with language once again appears and it shows Peirce's sensible perspective on striving for (or at least facilitating) proximity between reading, writing and arithmetic (the "three R's")

His concern with linguistic issues, as previously pointed out, cause Peirce to suggest that the child be allowed to make free associations in the initial phase of learning with respect to nomenclature to be used with each numeral:

> "*Many children will learn the names of numbers, and even apply those names pretty accurately, without having the slightest idea of what number is. This should not discourage the teachers. Such children learn by first acquiring the use of a word, a phrase, and then, long after, getting some glimmer of what it means. If it were not for this, formulas would not have the vogue they have - How many of those who talk of the law of supply and demand have any idea what that law is, further than that it regulates prices by the relation between wants and stocks of goods? /.../ Pay no attention to the ordinary names of numbers above nine. The child will learn those for himself. But in learning arithmetic the strict systematic character of numeration must be proeminent. Therefore, call ten, onety; eleven, onety-one; twelve, onety-two; thirteen, onety-three; twenty, twoty, etc.*"

The minimal treatment given to theory - even in his initial presentation - seems to be credited to his father's teachings. The influence of Benjamin Peirce in the mathematical work of his son is clear. In addition to the reference to Benjamin in the beginning of manuscript 179, to which we made reference here, there is a brief but significant reference to his father in the essay "*The Essence of Mathematics*", published in 1902: "It was Benjamin Peirce, whose son I boast myself, that in 1870 first defined mathematics as 'the science which draws necessary conclusions'. /…/ and my father's definition is in so far correct that is impossible to reason necessarily concerning anything else than a pure hypothesis."[26]. In another manuscript (MS. 905), he writes about his father's influence on his work in arithmetics:

> "*My father, then, was the leading mathematician of the country in his day - a mathematician of the school of Bowditch, Lagrange, Laplace, Gauss, and Jacobi - a man of enormous energy, mental and physical, both for the instant gathering of all his powers and for long-sustained work; while at the same time he was endowed with exceptional delicacy of sensation, both sensous and sentimental. But his pulse beat only sixty times in a [minute] and I never perceived any symptom of its being accelerated in the feats of strength, agility, and skill of which he was fond, although I have repeatedly seen him save his life by a hair-breadth; and his judgment was always sane and eminently cool. Without apearing to be so, he was extremely attentive to my training when I was a child, and*

> *specially insisted upon my being taught mathematics according to his directions. He positively forbade my being taught what was then, in this country, miscalled 'Intellectual Arithmetic', meaning skill in instantaneously solving problems of arithmetics in one's head. In this as in other respects I think he underrated the importance of the powers of dealing with individual men to those of dealing with ideas and with objects entirely governed by exactly comprehensible ideas, with the result that I am today so destitute of tact and discretion that I cannot trust myself to transact the simplest matter of business that is not tied down to rigid forms. He insisted that my instruction in arithmetic should be limited to exhibiting the working of an example or two under each rule and being set to do other 'sums' and to having my mistakes pointed out for my correction. He preferred that I should myself be led to draw up the rules for myself, and quite forbade that I should be informed of the reason of my rule. Thus he showed me, himself, how to use a table of logarithms, and showed that in a couple of cases the sum of the logarithms was the logarithm of the product, but refused to explain why this should be or to direct me to explanation of the phenomenon. That, he said, I must find out for myself, as I ultimately did in a more general way than that in which it is usually stated.*"

4. Some final considerations

A brief bibliographical review[27] seems sufficient to attest to the potential of Peirce's work for education, and in particular, mathematics education. There are similarities between Peircean theories and others that should be studied. There remains the possibility of taking advantage of Peircean pragmatism and his theory of signs as a philosophical foundation for mathematics education. Two of the publications reviewed (Strom, Kemeny & Lehrer, 1999 and Cassidy, 1982) attest to how useful Peirce's philosophical work can be as a methodological parameter. These possibilities, however, should not situate Peirce as a philosopher of education. The few texts he wrote on the subject are based more on his experience as a teacher and student and on the common sense of the thinker in sync with his time and with the possibilities of the future, and are not, as might be thought[28], an attempt to establish a philosophy of education. Peirce's thoughts on education are at best fragmentary.

We have to consider that Charles Peirce experienced almost every form of education. His father first gave him a home education and further allowed him to attend private schools "whose relative lack of rigorousness must have been a pleasant surprise". He also was student, teacher and researcher in important American institutions. We must recognize the well-known influence Peirce had on Dewey's philosophy, one of the most important and influential American educational philosophers. McCarthy (1971) points out that, in spite of all this, Peirce was curiously and uncharacteristically silent on educational questions. McCarthy analyses some documents collected by editors - at a time when most of Peirce's essays were not yet published - looking for evidence of a possible Peircean educational philosophy. But the author states clearly that, just because a man's thinking had powerful implications for education, it does not, of course, mean that he was an educational philosopher.

McCarthy uses as his main references a letter from Peirce to Daniel Coit Gilman (SW, pp. 325-330), a short paper published in the ***Educational Review*** in 1898 (SW, p. 338-341) and his assignment for the ***Century Dictionary*** published in 1889 for which he was one of the editors (SW, pp.332-335). McCarthy's paper was published in 1971, six years before Eisele published ***The New Elements of Mathematics by C. S. Peirce***, and although McCarthy considers the contribution Peirce made to the methods of teaching mathematics as "*evident from a series of manuscripts he prepared for three mathematics texts which were never published*", no more detailed references on this issue are presented. However, in discussing some possible further research on Peirce's mathematical works, he points out: "*Unfortunately, the reader who is not thoroughly grounded in mathematics is able to derive from these manuscripts only that Peirce had a sound concept of the order in which different types of mathematics ought to be taught, and that his schemes for scrapping rote learning and to involve the learner's imagination seem to antecipate the methods of the 'New Math'.*" Nowadays we can analyse the mathematical writings from a distinct perspective for they are all available in Eisele's edition, and also McCarthy's citation about New Math - which he seems to approve - can be considered in a quite different way (see, for instance, Klein, 1970).

Peirce's remarks on education in the papers used by McCarthy - currently published under the label "Science and Education" in Wiener's 1958 edition - focuses, in short, on Peirce's points of view which are not founded - or not explicitly founded - in his philosophy. There are no traces of pragmatism or of Peircean theory of signs when, in the letter to Gilman, Peirce exposed his thoughts on the organization and administration of an academic departament. But an educational concern can be detected there:

> "*/.../ the professor's objectives ought to be to let the pupil as much into the interior of the scientific way of thinking as possible, and for that purpose he should make his lecture experiments resemble real ones as much as possible, and he should avoid those exhibitions of natural magic which impress the mind with a totally perverted idea of science*" (SW, p. 326)

However, the implementation of an action-guided methodology within the laboratory in an inquiry approach - whose clear intention is to approximate as closely as possible the preparation of future physicists' with real practice and conditions, sharing responsibilities - is to be used only with the special students. The methods to be used with "general" students are lessons and lectures, but also this approach has new elements under consideration: beyond this treatment which involves lecturing, pupils ought to be exposed to the moral and logical lessons of physics, instructed as to the purposes, ideas, methods and life of the physicist. The main laws of physics must be taught to them in all of its possible applications.

> "*In regard to the system of instruction, the special pupils would give little trouble. They should be apprentices in establishment, above all. They should be left to work out the mathematics of practical problems in order that their mathematics might not be up in the air; they should also be made to study out new methods and make designs for new instruments, the instructor measuring their strength. /.../ Some of the merits of this method are that from the first the pupil feels himself an apprentice - a learner but yet a real worker; he is introduced to a great and important investigation and of this investigation he has a necessary part to do, he is not working for practice merely. /.../ The method with general students, in my opinion, is a more difficult problem than that with special students. For them are lessons. A lesson should be neither a recitation nor a lecture but something like a mixture of the two.*" (SW, pp 328-9)

Peirce's definition and remarks on the function of a university are also made clear in his definition in the ***Century Dictionary***, which is remarkable in that it makes not the slightest allowance for the function of instruction. McCarthy tells us that the other editors wrote to him that they conceived of the university as an institution for instruction. Peirce replied:

> "*If they have any such notion they were grievously mistaken, that a university had not and never had had anything to do with instruction and that until we got over this idea we should not have any university in this country*". (p. 10)

In his text, however, we find a remark on instruction being only a necessary means to the main purpose and function of a university: the production of knowledge. Also in this text Peirce deplored the tendency to evaluate professorial contributions in economic terms rather than in terms of theoretical research and affirms that universities seem to proclaim to its students that their individual well-being is its only aim. Peirce recognises himself as possibly guilty on this account, pointing out explicitly, for the first time in his "educational essays", the pragmatic approach:

> "*I am not guiltless in this matter myself, for in my youth I wrote some articles to uphold a doctrine I called Pragmatism, namely, that the meaning and essence of very conception lies in the application that is to be made of it. That is all very well, when*

> *properly understood. I do not intend to recant it. But the question arises, what is the ultimate application; at that time I seem to have been inclined to subordinate the conception to the act, knowing to doing. Subsequent experience of life taught me that the only thing desirable without a reason to being so, is to render ideas and things reasonable.*" (SW, p.332)

But this *mea culpa* is not enough to conceive these remarks on educational issues as founded on pragmatism or other aspects of Peircean theories. In realizing so, we strongly agree with McCarthy in saying that it would be possible and interesting - although probably a monumental and thankless labor - to elaborate in full detail what Peirce might have proposed as a Peircean philosophy of education. But, obviously, he did not do it himself.

Peirce's mathematical work – briefly presented in this paper – shows another side of the author that we feel has been neglected up to now by researchers in mathematics education. These manuscripts show inventiveness in the use of nomenclature, creativity in approach, and innovation. In particular, the manuscripts referring to his work in elementary mathematics (arithmetics, studied here) allow an analysis of Peirce's thinking about mathematics education. The emphasis on the link between language structure and the teaching of mathematics, the exploration of algorithms, the care taken to establish dialogues between diverse areas of knowledge (an initiative that has been given priority nowadays in proposals for education - interdisciplinarity), among other elements that await more detailed analysis - all these reinforce the potential of investing in Peirce. Mathematics education is not the only field that appears to have neglected Peirce's work; history of mathematics (and consequently, the history of mathematics education) seem to have overlooked it as well.

Finally, we hope the reader understands the need for the frequent and at times extensive quotes and notes in an article of this nature. Faithful documentation, respect for the authors' style (Peirce's, in particular), and a concern for the flow of the article were reasons we felt it necessary to preserve parts of the original texts and include the notes.

5. Notes

(1) As, for instance, Ernest (1991) tries to develop in his social-constructivist approach basically using Lakatos' and Wittgenstein's philosophies. As pointed out by Hanna (1994) mathematics educators have a special fascination with Lakatos' work, specifically with his *Proof and Refutations* (1976). Lakatos' falibilism, or his theory denying absolute mathematical truth, although often misinterpreted, is the main cause of that fascination (Garnica, 1996a).

(2) This affirmation - slightly romantic - about Peirce as a teacher will be better analyzed in McCarthy's paper (1971).

(3) Hartshorne and Weiss edited volumes 1 to 6. Arthur W. Burke edited volumes 7 and 8.

(4) In presenting Peirce's arithmetical texts we will follow Eisele's references in NE1. Manuscript (MS) identification numbers are related to Peirce's original Arithmetic as available in the Charles S. Peirce Collection at Houghton Library, Harvard.
(5) Actually, manuscript 189 (MS 189), *Lydia Peirce's Primary Arithmetic* has two distinct versions. The first, a reduced one, has an introduction and a presentation of the counting system (until the discussion of children's rhymes). The second version, more complete, presents again (with some new remarks) the introduction and develops some algorithms. Both versions must be seen as complementary.
(6) Not only in MS 182 as the editor points out.
(7) Based on Galton's works, Peirce considered those ghosts which usually inhabit human minds. Education in general and teachers in particular must be attentive in order to eliminate "hallucinatory imagination". Example of this hallucinatory conduct related to arithmetic issues can be detected in students who cannot conceive "number" with no colors, shapes or sizes: *"But there are others, who without localizing the objects of their imagination, still cannot think of an abstract number without the accompaniment of colors and shapes which have no intrinsic connection with the number. These persons get into the habit of thinking of each number in connection with constant fantastic shapes. Galton, in his Inquiries into Human Faculty (Macmillan, 1883, ...) has given many examples of this."*
(8) Lydia and Benjamin (who will soon appear in our story) are Christian names in the Peirce family.
(9) Peirce gives a lot of examples of these children's rhymes: *"Eeny, Meeny, Mony, Méye, Tusca, Rora, Bonas, Try, Cabell, Broke a well, Wee, Woe, Whack!"; "Peek, a Doorway, Tries, What wore he, Punchy, Switches, Caspar Dory, Ash-pan, navy, Dash them, Gravy, do you knock'em, Down!"* and *"One o'you, you are a, trickier, Ann, Phil I see, Fol I see, Nicholas John, Queevy, Quavy, Join the navy, Sting all'em strangle'em, Buck!"*. Nowadays, American children don't seem to know these rhymes, although they use a version slightly modified of the first presented in this note.
(10) In ancient times, Peirce tells the children, *"thousands of years ago before our grandfather's great grandfather's great great great grandfather's people had learned to build houses or do anything but fight and hunt and cook a little, only a few men knew to count"* (NE1, p.6-7). And when these people wanted a jury of twelve man and they had more than this to choose from, they count to thirteen and send away the thirteenth man and they would go on in this way until there was no longer a thirteenth man. In a letter to W.W. Newell, Peirce abductively conjectures: *"I have a beautiful theory. All it needs is some facts to support it, of which at present it is almost entirely destitute /.../ It is that in ancient times and the ages long gone by, men preferred a jury of 12 /.../ and used to count up to 13 to throw out extra candidates /.../. Now the only facts to support this that I have so far are, 1st, that the number 13 is widely associated with the idea of severance; 2nd, that our childhood's counting rhymes (as well as I remember) counted up to 13"* (NE1, p.7).
(11) Obviously Peirce's concern about tradition is not accidental or fortuitous. He defines it: *"Do you know what a tradition is? It is anything that older people have commonly taught to younger people for how long nobody knows".*

(12) Lydia says: *"Benjie, show me your right hand. Barbara, show me your right hand. Good, you both know which your right hand is. If you had not known, that would have been the first thing to learn. Now each of you hold out the right hand with the palm up. That is the palm. Now put the tip of the little finger of the left hand down upon the palm of the right hand and say, 'One'. Good! Now put the tip of the next finger of the left hand down upon the right palm along with the little finger, and say, 'Two'. /.../ Do it, now, again! Now again! That is your first lesson. Do it many times today and tomorrow; and when you have learned this well, we will go on to the other numbers."*

(13) Peirce shows particular examples to start the discussion about a concept or algorithm. In modern notation, using the function $f(x)=[x]$ (where $[x]=x$ if $x \in Z$ and $[x]$ is equal to the integer part of x if $x \in Z$); we could display it by saying that the quarter square of n, $n \in Z$, is $[n/2].(n - [n/2])$. Thus, the quarter square of 9 is 4.5=20; of 17 is 8.9=72, of 6 is 3.3=9, and so on.

(14) *"/.../ you must remember that such devices are merely [to] aid you in learning the multiplication table and to enliven the task a little, not to serve instead of a knowledge of the multiplication table. That must be learned so that you can recognize a letter of the alphabet no quicker than you remember the product of two numbers less than 10."*

(15) The quarter square algorithm also works well for numbers bigger than 10, but in such cases, we must develop calculations ("long multiplication"). Thus Peirce's care in discussing this special theme in a different item of his writings seems natural.

(16) 365
$\underline{127}$
3 (*"Multiply each figure of the multiplicand by every figure of the multiplier and set down the product so that its unit place shall be the number of places to the left of the units place of the factors which is the sum of the numbers of places by which the two figures multiplied are to the left of the units' place. Thus, 1 times 3 is 3,; the 1 is 2 places to the left of the units' place, the 3 is 2 places to the left of the units. 2 and 2 make 4. So we set down the 3, the product, 4 places to the left of the units /.../"*).

(17) With respect to the "position" of the solution: *" 'How much is 6 times 7?' '42' 'And how much is 6 times 70?' '420' 'And how much is 60 times 70?' 'Ten times as much, 4200' 'Well, that makes the whole thing clear does it not?' 'I must think over that', said Benjie."*

With respect to the need to sum the products to get the answer: *"'Why, you see 2 times 7 and 3 more times 7 make, in all, 5 times 7, don't they?' 'Yes' 'So, 7 times 2 and 7 times 3 make 7 times 5, don't they?' 'Yes' /.../ 'And 100 times 300 and 100 times 65 make 100 times 365. And 27 times 300 and 27 times 65 make 27 times 365. So:*

100 times 300
and 100 times 65
and 27 times 300
and 27 times 65 make 127 times 365." The process is quite analogous to that recommended nowadays for the application of distributiveness. This property is certainly the essence of the algorithm.

(18) *"Now begin at the right, and say 7 times 5 make 35. Set down the 5 in the units place and carry the 3 to the tens place /.../"*

(19) *"But the second way [the algorithm with the carried numbers formally written down] is the best way; because this third way is too hard. You are apt to make mistakes. But if there are only two figures, it is a good way."*
(20) In the manuscript 1893, Peirce analyzes the treatment given to addition and subtraction in some didactic texts. He states: *"Teach one thing at a time, is that the most of them forget. But slight preparatory hints of what is coming without special teaching is permissible and recommendable"*. Following that is a list of the "terminology" he would use to teach these operations. Manuscript 1546 is "the" reference, because of its excellence, on the consultancies and analyses done by Peirce. In it is a list of 44 texts (*"Arithmetics now, or late, used in American Schools"*) followed by another list of the works consulted to elaborate his opus, the majority classical European texts, mostly on Mathematics and the History of Mathematics from the fifteenth and sixteenth centuries.
(21) *"There is nothing more instructive for children in many ways than cards bearing the successive numbers from 1 to 100. Each number should be expressed in Arabic figures, below; and above should be that number of red spots. These dots may be arranged so as to show the factors of the number, or if it is a prime to show that it is one more or less than a multiple of six. For if the arrangement should be remembered, which is not to be desired, it will, at least, recall a fact of value."*
(22) On the abacus (*"a frame with seven parallel wires, and nine beads in each wire. All the beads on one wire are the same color"*) counting is done in groups of ten. Thus, the children initially have the numbers in groups of ten. In this way, 221 is Two ten tens two ten one; 116 is Ten tens ten six; 211 is Two ten tens ten one, etc. The advice for the sequence for the activity with the abacus (which is an implicit discussion about the need to take care with the language) is in the following quote: *"The counting of the marbles is to be practised until the pupils thoroughly understand it, and are perfectly familiar with the numbers. Teacher (holding up a glove): Is this a shoe? All: No; Teacher: No: it is not because it is not meant to walk in. What is it? All: A glove. Teacher: Yes. It is meant to wear on the hands. It is called a glove. It is a thing meant to wear on the hands, with a place for each finger. Glove is its name. It is much more convenient to say give me a pair of gloves, than to say give me a pair of things to wear on my hands with a place for each finger. /.../ Some of the numbers have easy names. Two tens is twenty. /.../ Ten ten tens is a thousand. Let us count by bags of ten. /.../ Let us count by bags of ten tens /.../"*.
(23) In the text on arithmetics, cards with the Hindu-Arabic Numerals on one side and nothing on the other are used for the relation between quantities (presented in sets of points, drawings, and marks) and their graphic representation in the Arabic system: *"There can be little harm in the association with number of the Arabic figure, or figures, which express it. /.../ for several reasons it will be best to encourage the association with a number of the Arabic expression of it."* (MS. 179) The frequent utilization of graphic representations and various drawings lead some specialists to see in this a desire of Peirce to stimulate the students to move closer to a topological way of thinking, given the importance that the author attributes to topology in his *Geometry*. Nonetheless, nothing in our reading of the manuscripts leads us to agree with this speculation.

(24) At the beginning of manuscript 189 (first version), the advice: *"(The children are not supposed to be able as yet to read. Nevertheless, they will need copies of this book, as will appear, soon. The first lessons are to be read to them by the teacher, who must be provided with a separate copy of the book.)"*

(25) In the general plan for his ***Primary Arithmetic***, manuscript 180, Peirce points out: *"I. The first ten numbers and their succession to be taught. (The Arabic figures to be shown but not insisted on.) Their use in counting. Exercises in counting objects in the room, with use of the 'Number Cards'. Counting various figures. II. Higher numeration, with Arabic figures. False names to be used first, with a view of keeping regularities of language in the background till the Arabic system is understood. Then the usual names to be introduced. III Exercises in counting considerable numbers, up to a thousand with rapidity and accuracy. IV. Counting by tens. V. Counting by fives. VI. Counting by twos. VII. Counting by nines. VIII. Counting by eights. IX. Counting by fours. X. Counting by sixes. XI. Counting by three. XII. Counting by sevens. In all these lessons the number-cards are to be used at first. Afterwards, coffee beans. The drill is to be carried so far that given any number under ten, the pupil immediately proceeds from that with perfect fluency, adding successive 1s, 2s, 3s, 4s, 5s, 6s, 7s, 8s, 9s, 10s, etc. to 101. This drill is the foundation of all facility in arithmetic. Competition and prizes. XIII. Sums in addition of two numbers done in the head, and expressed concretely. XIV Adding columns. These are gradually lengthened until fifty figures. Minute attention to all the details of the methods. XV. Simple subtraction. XVI. Subtraction taught with the abacus. XVII. Multiplication."*

(26) From this passage, Benjamin Peirce's profound influence on his son can be deduced. It is in these considerations about the nature of mathematics that Peirce bases his theory about forms of reasoning, re-visiting the Greeks, developing the concepts of induction, deduction, and abduction: *"To discover that we know through the combination of three fundamental forms of inference is to take a necessary but not fully sufficient step toward the development of a scientific method. The three kinds of argument have been known and explained since the times of the Greeks. /.../ above all, I stress the importance of the function of abduction, of hypothesis. By emphasizing against the Cartesian tradition, that all our knowledge has a hypothetical basis, on the other hand I highlight its intrinsic fallibility but on the other I proclaim the need resolutely to put abduction in the control room of cognitive process in general and above all in the scientific process, for its only by means of new hypotheses and bolder abductions, that we can discover new truth, however approximate and provisional; its only by means of new hypotheses that we can widen our vision of the real and discover new avenues of experience, propose new material for the test bench of experimentation"* (quoted in ECO and SEBEOK, 1983)

(27) Although the complete bibliographical review we have done to give foundation to this study is not presented in this paper, all the references are listed below.

(28) Approaches and terminology can cause confusion, such as in the case of the so-called Peircean phenomenology, mistakenly interpreted as a proximity of Peirce to Husserl's phenomenology (consult Ransdell, 1989, for more information on this). Some liberties are also taken on the link between pragmatism and Peirce's sign theory. According to Cooper, (1967), *"Peirce's sign theory is not strictly necessary to his pragmatism, which can be taken* *as a logical role related to the conceptions of inquiry and inference. But much of the richness of Peirce's thought, his creative and largely independent construction of pragmatism as a general philosophic frame, would thereby be lost as would some of the relation to Dewey's later sign theory"* (p. 12). We reiterate that, in our study, we only raise the possibility of constituting rigorous philosophy thought about education and mathematics education based on Peirce's work.

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