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THE DIVIDED LINE AND THE GOLDEN MEAN

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Resumo

No presente trabalho faz-se uma revisão da nossa interpretação da estrutura matemática da Linha Dividida de Platão e a estende para a Linha Duplamente Dividida, mostrando assim o alcance da teoria. Então, explicamos detalhadamente em que maneira a solução de Eudoxo para o problema de incomensurabilidade é relacionada à teoria de *lógos*, o que faz possível o desenvolvimento de Linhas Divididas com componentes "irracionais". Finalmente, linhas Divididas não-aritméticas foram desenvolvidos e Platão usou-las para modelar contextos não-matemáticos.

Palavras-chave: Matemática e Filosofia; Linha Dividida; Incomensurabilidade; Teoria de Proporção; Platão; Eudoxo.

Abstract

The present paper reviews our interpretation of the mathematical structure of Plato's Divided Line and extends it to the Doubly Divided Line, thereby showing how pervasive the doctrine can be. We then explain in detail how Eudoxus's solution of the problem of incommensurability is related to *lógos* theory, which makes possible the development of Divided Lines with "irrational" components. Finally, non-arithmetical Divided Lines were developed, which Plato used to model non-mathematical contexts.

Keywords: Mathematics and Philosophy; Divided Line; Incommensurability; Theory of Proportions; Plato; Eudoxus.

In Book VI of *The Republic*, Plato sets out his doctrine of the Divided Line, according to which there is a hierarchical set of levels of reality and modes of apprehension appropriate to each. This hierarchy can be set out in the following manner, in which the levels ascend from left to right:

Images / Cosmological Forms / Mathematical Forms / Transcendental Ideas.

It is not our purpose here to investigate the nature of each of these different levels.[1] Rather, we wish to point out a curiously neglected aspect of Plato's treatment of the Divided Line. Line segments were used in Antiquity to stand for numbers and, eventually, incommensurable magnitudes. Thus, it is, to say the least, very curious that Plato has them represent non-mathematical entities. Yet, he does so quite decidedly. In Cornford's (1969, p. 224) translation, we have:

> Now take a line divided into two unequal parts, one to *represent* the visible order, the other the intelligible; and divide each part again in the same proportion, *symbolizing* degrees of comparative clearness or obscurity. [Emphasis added.]

We seem to have here a clear case of consciously analogical reasoning in an epoch in which "analogy" meant proportion. We wish to suggest herein how Plato may have moved from the original idea of proportion among positive whole numbers to a more comprehensive notion of analogy by way of Eudoxus' theory of proportion and the application of the "Golden Mean" to the structure of the Divided Line.

In order to fulfill our stated purpose, we will set out in some detail our previous reconstruction of the mathematical structure of the Divided Line[2] and extend it to the Doubly Divided Line. This will provide the backdrop against which the extension to the case of more General Lines will be made. This extension will be presented, however, only after carefully investigating Eudoxus' theory of proportion and comparing it to the older Pythagorean theory. We will avoid anachronistic interpretations by viewing Plato's understanding of "analogy" not in its full blown modern acceptation, but as revealing the mathematical structure of apparently non-mathematical situations. Finally, we will conclude by showing how Plato's new understanding of analogy led him to a more profound interpretation of the Pythagorean presupposition that the intelligibility of the world is due to its mathematical structure.

The Divided Line

According to Plato's account in Book VI of *The Republic*, a line (*i.e.*, a line segment) is divided into two unequal parts and each of these parts is then divided in the same ratio. The result is a line segment with four parts, which we may represent iconographically as $\underline{x\ /\ y\ /\ z\ /\ t}$. From the stated conditions, we have $x+y:z+t::x:y::z:t$. It has long been realized that these conditions imply that the two middle segments are equal. It is easier for us to see this if, instead of the ratio-proportion language used by the ancient Greeks, we use the equivalent in fraction-equality language. Thus, we have $(x+y)/(z+t) = x/y = z/t$. By cross multiplication, we obtain the following equations:

[1] See further Erickson and Fossa (to appear).
[2] See Erickson and Fossa (to appear) or Erickson and Fossa (2001).

$$xy+y^2 = xz+xt$$
and $$xt = yz.$$
Substituting for xt in the first equation gives
$$xy+y^2 = xz+yz$$
$$(x+y)y = (x+y)z$$
$$y = z.$$
In consequence, the Divided Line reduces to $\underline{x\ /\ y\ /\ y\ /\ z}$, where $x+y{:}y+z{::}x{:}y{::}y{:}z$; that is, where $(x+y)/(y+z) = x/y = y/z$ and, thus, y is the geometric mean between the two extremes x and z. When, however, we impose the further condition that the variables represent natural numbers (positive integers), we find that the Divided Line has a much more definite mathematical structure. Thus, let $n{:}m$, where m and n are relatively prime positive integers (n/m is a fraction in lowest terms), be the ratio into which the segment was originally divided, so that $(x+y)/(y+z) = x/y = y/z = n/m$. Proceeding as above, we obtain
$$xm = yn$$
and $$ym = zn.$$
Hence, $$y = zn/m$$
and $$x = zn^2/m^2.$$
Now, in order for x to be an integer, m^2 must divide evenly into zn^2. But m and n are relatively prime and, thus, m^2 must divide evenly into z. Thus, $z = km^2$, for some integer k. Substituting this value for z in the above equations, we find that $x = kn^2$ and $y = knm$. Thus, the Divided Line reduces to:
$$\underline{kn^2\ /\ knm\ /\ knm\ /\ km^2},$$
or, factoring out the k common to all terms:
$$k(\underline{n^2\ /\ nm\ /\ nm\ /\ m^2}).$$
When k=1, we say that the Divided Line is Primitive; otherwise, it is Composite. Thus, a Primitive Divided Line corresponds exactly to the first part of Plato's Theorem[3], for the middle term is the integral geometric mean between two square numbers.

In the passage alluded to above, Plato divides the Line into two unequal parts. This is necessary for the philosophical doctrine of the ontological and epistemological hierarchies that he is propounding in that passage. From a mathematical point of view, however, the structure of the Divided Line is still satisfied when it is divided into equal parts. In the case that it is Primitive, we have $\underline{1\ /\ 1\ /\ 1\ /\ 1}$, which we will call the Monadic Line. It will be important in what follows. When, however, the division is into two unequal parts, we will always follow the convention of writing the large extreme first; we also write ratios $n{:}m$ with $n>m$.

The mathematical structure of the Divided Line is characterized by two theorems. The first describes a process of generating new Divided Lines and the second guarantees the completeness of the process.

> **The Algorithm of the Divided Line**. (i) The sum of the elements of a Primitive Divided Line is to the sum of the large extreme and the mean as this sum is to the large extreme. (ii) The sum of the elements of a

[3] That the geometric mean between two perfect squares is integral and that the two mean proportionals between two perfect cubes are integral is refered to by Nicomachus (1938, p. 272-273) as " a certain Platonic theorem".

> Primitive Divided Line is to the sum of the small extreme and the mean as this sum is to the small extreme. In both cases, the new Divided Line is Primitive.

To see the first part of this theorem, let $\underline{n^2 / nm / nm / m^2}$ be a Primitive Divided Line, so that m and n are relatively prime, with $n>m$. The Algorithm asserts that

$$\underline{n^2+nm+nm+m^2 / n^2+nm / n^2+nm / n^2}$$

is a Primitive Divided Line. Simplifying the additions, we have

$$\underline{(n+m)^2 / (n+m)n / (n+m)n / n^2}.$$

Clearly, $(n+m)^2{:}n(n+m){::}n(n+m){:}n^2$, the ratio being $(n+m){:}n$. Thus, we have only to show that the Line is Primitive, that is, that $n+m$ and n are relatively prime. This is an easy exercise. Suppose that some whole number, d, divides evenly into both $n+m$ and n. Then, $n+m=k_1d$ and $n=k_2d$. Substituting the latter expression into the former, we have $k_2d+m=k_1d$ and, thus, $m=k_1d-k_2d=(k_1-k_2)d$. That is, d also divides evenly into m. But, since, m and n are relatively prime, d must be 1. Hence, the new Line is Primitive.

Part (ii) of the Algorithm generates the Divided Line

$$\underline{(n+m)^2 / (n+m)m / (n+m)m / m^2}.$$

The proof that this Line is Primitive is entirely analogous to that of part (i).

We now show that the Algorithm is complete.

> **The Theorem of the Divided Line**. The Algorithm of the Divided Line generates all and only Primitive Divided Lines from the Monadic Line $\underline{1 / 1 / 1 / 1}$.

In order to prove this Theorem, it will clearly be sufficient to show how, given any Primitive Divided Line, we can trace a path backward to the Monadic Line, which path, when reversed, will generate the given Line by successive applications of the Algorithm. To this end, let

$$\underline{n^2 / nm / nm / m^2} \qquad (*)$$

be a Primitive Divided Line, so that m and n are relatively prime, with $n>m$. We now ask ourselves what Line could generate (*). Since the Algorithm prescribes that one of the new extremes will be a sum, we now do the inverse operation of subtraction, namely, $n-m$. There are two cases to consider[4], for $n-m$ may be greater than or less than m. We first investigate case (a): $n-m>m$. The Line that generates (*) would then have $(n-m)^2$ as the large extreme and m^2 as the small extreme. Hence, we would have

[4] Clearly, $n-m=m$ is impossible, since in this case n would be $2m$ and, thus, n and m would not be relatively prime.

$$\underline{(n\text{-}m)^2 \,/\, (n\text{-}m)m \,/\, (n\text{-}m)m \,/\, m^2} \qquad (**),$$

which is clearly a Divided Line. Since it is the small extreme, m^2, of (**) that appears in (*), we apply part (ii) of the Algorithm to (**) to verify whether it generates (*):

$$\underline{(n\text{-}m)^2+(n\text{-}m)m+(n\text{-}m)m+m^2 \,/\, (n\text{-}m)m+m^2 \,/\, (n\text{-}m)m+m^2 \,/\, m^2}.$$

But, $(n\text{-}m)^2+(n\text{-}m)m+(n\text{-}m)m+m^2 = n^2\text{-}2nm+m^2+nm\text{-}m^2+nm\text{-}m^2+m^2 = n^2$. Further, $(n\text{-}m)m+m^2 = nm\text{-}m^2+m^2 = nm$. Thus, applying the Algorithm to (**) does indeed produce (*).

Now, if any whole number, d, divides evenly into $n\text{-}m$ and also into m, it will divide evenly into their sum, $n\text{-}m+m$, which is just n. But, since n and m are relatively prime, d=1 and, consequently, $n\text{-}m$ and m are relatively prime. Thus, (**) is Primitive. We have, therefore, found a Primitive Divided Line, (**), that generates (*).

We now investigate case (b): $n\text{-}m<m$. In this case, the Line that generates (*) would have $(n\text{-}m)^2$ as the small extreme and m^2 as the large extreme. Hence, we would have

$$\underline{m^2 \,/\, m(n\text{-}m) \,/\, m(n\text{-}m) \,/\, (n\text{-}m)^2} \qquad (***).$$

It is now the large extreme, m^2, of (***) that appears in (*), so we apply part (i) of the Algorithm to (***). In exactly the same way as in case (a), applying the Algorithm to (***) results in (*). In fact, (***) is, so to speak, the mirror image of (**), so the algebra is identical. Again, in exactly the same way as in the preceding case, it is shown that (***) is Primitive.

What we have shown so far is that, given any Primitive Divided Line, we can find a new Primitive Divided Line that generates the given Line when the Algorithm is applied to it. One of the extremes of the new Line will be the same as one of the extremes in the given Line; the other extreme of the new Line will be less than the other extreme of the given Line. Thus, by reiterating the process, we obtain new Lines in which the extremes get less and less. Eventually, since there is only a finite number of whole numbers between n and 1, we come upon a new Line in which the small extreme is 1.[5] That is, we obtain the line $\underline{k^2 \,/\, k \,/\, k \,/\, 1}$. But this Line is clearly generated by $\underline{(k\text{-}1)^2 \,/\, k\text{-}1 \,/\, k\text{-}1 \,/\, 1}$. Continuing in this manner, we obtain the Monadic Line $\underline{1 \,/\, 1 \,/\, 1 \,/\, 1}$, in k-1 steps. Thus, we have proven the Theorem of the Divided Line.[6]

The Theorem of the Divided Line is quite remarkable. Almost as impressive, however, is to actually start with the Monadic Line and calculate new Divided Lines, using the Algorithm of the Divided Line[7]. At each level, twice as many new Divided Lines are generated as at the previous level, so that one gets the impression of an unfolding cascade,

[5] This is, of course, equivalent to Euclid's Algorithm for finding the G.C.D.

[6] For the convenience of the reader, we have freely used algebraic symbolism to express the argument. That, however, all the required mathematics was known at Plato's time is shown in Fossa and Erickson (to appear). There, it is also suggested how the above two theorems might have been demonstrated.

[7] We will look at some special cases below.

much in the same way, we may imagine, as Plotinus' successive emanations.[8] Be that as it may, we now turn to the second part of Plato's Theorem and Doubly Divided Lines.

The Doubly Divided Line

We now generalize the Divided Line in order to obtain the second half of Plato's Theorem. Thus, divide a line segment in two parts and divide each part in the same ratio. The result is the Divided Line $\underline{x / y / y / z}$. We now divide each of these four parts into the same ratio. On doing so, the two means, y, will, of course, be divided in the same way. Further, the $\underline{x / y}$ pair will be split into a Divided Line, thus having equal middle parts. The same applies to the $\underline{y / z}$ pair. In this way we obtain $\underline{a / b / b / c / b / c / c / d}$, where $a{:}b{::}b{:}c{::}c{:}d{::}a+b{:}b+c{::}b+c{:}c+d{::}a+b+b+c{:}b+c+c+d$.

We now ask under what conditions the Doubly Divided Line will be integrally valued. We may assume that the ratio of the division is $n{:}m$, where $n, m \in \mathbf{N}$ and $n{:}m$ is in lowest terms. Replacing the ratio-proportion language by fraction-equality language, we obtain:

$$a/b = n/m$$
$$b/c = n/m$$
$$\text{and} \qquad c/d = n/m.$$

Solving for the numerator on the left hand side of each equation gives:

$$a = bn/m$$
$$b = cn/m$$
$$\text{and} \qquad c = dn/m.$$

Substituting for b and c, we obtain:

$$a = cn^2/m^2$$
$$\text{and} \qquad a = dn^3/m^3.$$

Since n/m is in lowest terms, m^3 must be a factor of d, that is $d=km^3$ for some $k \in \mathbf{N}$. Thus, $a=kn^3$. We obtain c and b by substitution: $c = dn/m = (km^3)n/m = knm^2$ and $b = cn/m = (knm^2)n/m = kn^2m$. Therefore, any Doubly Divided Line with integral components will have the form:

$$k\ [\ \underline{n^3 / n^2m / n^2m / nm^2 / n^2m / nm^2 / nm^2 / m^3}\ ].$$

When $k=1$, we say the Doubly Divided Line is Primitive; otherwise, it is Composite. Thus, in any Primitive Doubly Divided Line, the extremes are relatively prime cubes. Further, we call n^2m the large mean, nm^2 the small mean, $\underline{n^3 / n^2m / n^2m / nm^2}$ the large component line and $\underline{n^2m / nm^2 / nm^2 / m^3}$ the small component line. Finally, we say that the means mediate the extremes and that they mediate primitively when the Line is Primitive.

We now generalize to this case the Algorithm of the Divided Line and the Theorem of the Divided Line.

> **The Algorithm of the Doubly Divided Line**. (i) The sum of the elements of a Primitive Doubly Divided Line is mediated primitively to

[8] For the relation between Plotinus and Plato, see Erickson and Fossa (to appear).

> the large extreme by the sum of the elements on the large component line and the sum of the large extreme with the large mean. (ii) The sum of the elements of a Primitive Doubly Divided Line is mediated primitively to the small extreme by the sum of the elements on the small component line and the sum of the small extreme with the small mean.

To see this for part (i), let $\underline{n^3 / n^2m / n^2m / nm^2 / n^2m / nm^2 / nm^2 / m^3}$ be a Primitive Doubly Divided Line and, thus, n and m are relatively prime. The sum of the component parts of this Line is $n^3+3n^2m+3nm^2+m^3$, which is just $(n+m)^3$. The sum of the elements on the large component line is $n^3+2n^2m+nm^2 = (n^2+2nm+m^2)n = (n+m)^2n$. Finally, the large extreme plus the large mean is n^3+n^2m, or $(n+m)n^2$. Fitting all these results together in the prescribed way we obtain

$$\underline{(n+m)^3 / (n+m)^2n / (n+m)^2n / (n+m)n^2 / (n+m)^2n / (n+m)n^2 / (n+m)n^2 / n^3}\ .$$

As we saw above, $n+m$ and n are relatively prime when n and m are relatively prime; therefore, the result is a Primitive Doubly Divided Line.

Part (ii) of the algorithm is similar. The two sums indicated are just $(n+m)^2m$ and $(n+m)m^2$. The new Primitive Doubly Divided Line is then:

$$\underline{(n+m)^3 / (n+m)^2m / (n+m)^2m / (n+m)m^2 / (n+m)^2m / (n+m)m^2 / (n+m)m^2 / m^3}\ .$$

Thus, the Algorithm of the Doubly Divided Line is established.

It should be clear that these two algorithms can be generalized and, thus, the mathematicians of the Academy were just a step away from the binomial theorem and Pascal's Triangle.

We now formulate the following theorem:

> **The Theorem of the Doubly Divided Line**. The Algorithm of the Doubly Divided Line generates all and only Primitive Doubly Divided Lines from the Doubly Monadic Line $\underline{1 / 1 / 1 / 1 / 1 / 1 / 1 / 1}$.

As in the case of the Theorem of the Divided Line, we start from any Primitive Doubly Divided Line and ask what Line, using the Algorithm, would generate it. Again, since the Algorithm proceeds by addition of the elements of the basic ratio n:m, we use the inverse operation of subtraction. Thus, given any Primitive Doubly Divided Line

$$(*) \quad \underline{n^3 / n^2m / n^2m / nm^2 / n^2m / nm^2 / nm^2 / m^3}\ ,$$

we have, by our convention, $n>m$ (and, of course, n and m, relatively prime). Consequently, $n-m$ is a positive whole number such that either $n-m$ is greater than m or $n-m$ is less than m.

For $n-m$ greater than m, we consider

$$(**) \quad \underline{(n-m)^3 / (n-m)^2m / (n-m)^2m / (n-m)m^2 / (n-m)^2m / (n-m)m^2 / (n-m)m^2 / m^3}\ .$$

Clearly, (**) is a Doubly Divided Line. Further, $n-m$ and m are relatively prime because n and m are relatively prime. Thus, (**) is also Primitive. Now we apply the Algorithm to (**) in order to verify that it generates (*). Since (*) keeps the small extreme from (**), part (ii) of the Algorithm is applicable. A bit of algebra shows that the sum of all the elements of (**) is just n^3. Thus, applying the Algorithm to (**) will give us a Line whose extremes are n^3 and m^3. The new large mean is the sum of the last four elements of (**), which, after another bit of algebra, is just n^2m. Finally, the new small mean is the sum of the small extreme of (**) with its small mean and this turns out to be nm^2. Thus, applying the Algorithm (part ii) to (**) does indeed produce (*).

For n-m less than m, we consider

(***) $\underline{m^3 / m^2(n-m) / m^2(n-m) / m(n-m)^2 / m^2(n-m) / m(n-m)^2 / m(n-m)^2 / (n-m)^3}$.

Since m^3 is now the large extreme in (***) and it is kept in (*), part (i) of the Algorithm is applicable. Further, since (***) is, so to say, the mirror image of (**), it turns out that the algebra is exactly the same in this case as in the previous one. Thus, applying the Algorithm (part i) to (***) produces (*).

In the same way as happened in the case of the simple Divided Line, by reiterating the process, we eventually reach a Doubly Divided Line whose small extreme is 1. From this, we can work back to the Doubly Monadic Line and the Theorem is thus established.

In the Divided Line, we have a perfect parallel between the Line and the proportion it gives raise to:

$$a / b / b / c$$

and $\quad a{:}b{::}b{:}c.$

This does not carry over in the Doubly Divided Line, since the Line has eight parts, but the proportion only six:

$$a / b / b / c / b / c / c / d$$

but $\quad a{:}b{::}b{:}c{::}c{:}d.$

Thus, it would be interesting to describe the Doubly Divided Line in terms of the Divided Line. This is easily done. The following theorem does so for Primitive Lines, but the generalization to any Line is almost immediate. We introduce the following terminology: the "roots" of a Primitive Line are the roots of its extremes (we may also call these the "generators" of the Line); the "primitive ratio" of a Primitive Line is the ratio of its roots, the larger to the smaller.

> **The Double Line Theorem**. In all Primitive Doubly Divided Lines, the large component is the Primitive Divided Line with the same roots as the Doubly Divided Line multiplied by the first term of its primitive ratio; the small component is same Divided Line multiplied by the second term of its primitive ratio.

The proof is straightforward. Let

$$\underline{n^3 / n^2m / n^2m / nm^2 / n^2m / nm^2 / nm^2 / m^3} ,$$

be a Primitive Doubly Divided Line. Thus, n and m, the roots of the Line, are relatively prime and $n{:}m$ is its primitive ratio. The Primitive Divided Line with the same roots is $\underline{n^2 / nm / nm / m^2}$. Thus, we have only to factor out n from the large component and m from the small component to prove the Theorem:

$$n(\underline{n^2 / nm / nm / m^2}) / (\underline{n^2 / nm / nm / m^2})m.$$

The mathematical structure of the Platonic doctrine of the Divided Line is thus seen to be a compelling and beautiful structure, based on ratios and proportions among positive whole numbers. We now turn our attention to Eudoxus.

Eudoxus' Accomplishment

The standard interpretation of Eudoxus' theory of proportion is that it incorporates the irrational numbers into the realm of mathematics and thereby saves mathematics from the "scandal" thrust upon it by the discovery of incommensurability. It is also frequently observed that his theory is virtually identical with Dedekind's construction of the irrationals in the nineteenth century. Now, all of this is so clearly anachronistic that certain caveats are often employed to attenuate the more ahistorical aspects of the interpretation, but even this attenuated standard interpretation is not really satisfactory.[9] Thus, we will now take a closer look at just exactly what Eudoxus' theory of proportions does and how it does it.

The first thing that should be noticed is the difference between the modern notion of irrational numbers and incommensurability. The irrationals are indeed an extension of the rationals, in which the basic arithmetical operations carry over. For the ancient Greeks, in contrast, incommensurability is a relation between *two* magnitudes: the relation that obtains when the two do not have a common measure. Now, that there should be incommensurables at all is a very surprising, nonintuitive result. It was also a disturbing result to the Pythagoreans because it undermined their mathematics and, thus, their whole philosophy. This is often expressed in the literature by the observation that incommensurability contradicts the Pythagorean axiom that "all is number". This, however, does not tell the whole story.

More careful renditions express the Pythagorean position as "all is number and harmony".[10] In order to fully comprehend the Pythagorean dictum, however, we must flesh out the meaning of "harmony". Later to be called music, harmony is ratio (λόγον) and proportion ('ανά λόγον, later written as a single word 'ανάλογον, "analogy"). The 'ανά here means "throughout" and has an intensifying quality. (Compare γιγνώσκω "know" and 'αναγιγνώσκω "know well".) Thus, a proportion is a sequence of terms in which a ratio (*lógos*) is carried over to the component pairs. It does not seem to have been noticed that, for the Pythagoreans, number itself is a *lógos*. Number is a collection of units[11] and is therefore measured by the unit. Thus, 7, for example, is thought of as 7:1 and can only exist as a number (a multiplicity) in virtue of its relation of *lógos* to the unit. In more complex ratios, the second term is thought of as a unit which either measures the first term integrally, as in 6:3 (a musical octave), or with the help of its parts, as in 4:3 (a musical fourth). The latter case is possible, because the unit (3) can be decomposed into subunits (1) that also measure the first term (4). In consequence, all ratio (*lógos*) is the relation of a first term measured by a second term, or a submultiple of the second term, and this then becomes the Pythagorean model for intelligibility.

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[9] Grattan-Guinness (1996) denies the purported near identity of Eudoxus' theory of proportion and Dedekind's theory of irrationals. He also denies that Eudoxus' theory is an extension of proportion theory to irrational numbers. Rather, he claims that Euclid set out two different proportion theories, one for integers and one for magnitudes, with similar structures.

[10] This is based on Aristotle's remark (*Metaphysics*, 986[a]): "... — since, then, all other things seemed in their whole nature to be modeled on numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number."

[11] Euclid, Book VII, Definition 2.

The preceding paragraph only hints at the Pythagoreans' sophisticated understanding of ratio. In particular, the unit, the numerical correlate to the Monad, the font of all being, is not a fixed quantity, but is anything that can be regarded as having an internal unity; that is, it is anything that can be thought without regard to its component parts.[12] Thus, as we have seen, 3 may be regarded as a unit that measures its multiples[13] and 4 and 3 are commensurable because they are measured by a common unit, 1. Number was often represented by line segments, but there was no set length for the unit line segment. This means that, for example, the side of a square of area 32 to the side of a square of area 18 is a musical fourth. The two sides are commensurable, having the side of a square of area 2 as their common unit measure. The sides in question are $4\sqrt{2}$ and $3\sqrt{2}$ and the unit is $\sqrt{2}$. Thus, even though all these are irrational magnitudes, we still have *lógos* in the aforementioned ratio.[14]

The trouble comes with such expressions as $4\sqrt{2}:3$, or, more simply, $\sqrt{2}:1$. The problem is that neither 1, nor any submultiple of 1 (that is, $1/n$, where n is a positive whole number), measures $\sqrt{2}$. Thus, $\sqrt{2}$ is 'άλογος, "irrational" and, by extension, unintelligible. Nevertheless, there are many examples of incommensurable magnitudes in mathematics and mathematics, we recall, is the model for true knowledge. The inescapable consequence of this situation – and this is the true scandal of incommensurability – is that it makes true knowledge gibberish. Further, even should we be able to make some kind of sense out of irrational relations, the demonstrations developed for rational relations would probably not carry over into the more general situation. But what makes mathematics mathematics is its method of verification by demonstration. So, once again, incommensurability struck a hard blow to one of the deep-seated tenets of the Pythagorean worldview.

What was needed then was some way of making incommensurables reasonable. It was exactly this that Eudoxus did with his theory of proportion. His first step was as bold as it was simple. He generalized the definition of *lógos* in the following way:

> A **ratio** is a sort of relation in respect of size between two magnitudes of the same kind. (Euclid, Book V, Definition 3.)

This definition has been criticized as being mathematically worthless since is unnecessary and vague.[15] But this criticism misses the whole point. *Lógos* had previously meant ratio between positive whole numbers; further, associated with this meaning was the assumption that *lógos* conferred intelligibility. Thus, it was quite a bold step to extend the meaning of *lógos* to relations that involved incommensurables because these were seen as '*álogos* or

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[12] Euclid, Book VII, Definition 1.

[13] Euclid's definition (Definition 20, Book VII) of "proportion" also makes this clear, for numbers are said to be in proportion when the first is the same multiple, part or parts of the second as the third is of the fourth. That is, when we use the second as a unit in relation to the first, we obtain the same *logos* as when we use the fourth as a unit in relation to the third.

[14] See Euclid, Book X, Definition 1.

[15] For more details, see Heath's commentary. Indeed, many later mathematicians were uncomfortable with the new theory; see Palmieri (2001).

irrational. But it was also a quite necessary step because, without it, the whole theory of proportions could not get started.

Then, Eudoxus set out the following conditions for four magnitudes to have the same ratio[16] ('εν τῷ αυτῷ λόγῳ), that is, to be proportional[17] ('ανάλογον):

> Let A, B, C, and D be four magnitudes and n and m be positive whole numbers. Then, A, B, C, and D are proportional if for every pair n, m, the following three conditions are satisfied:
>
> 1. $nA>mB \Rightarrow nC>mD$
> 2. $nA=mB \Rightarrow nC=mD$
> 3. $nA<mB \Rightarrow nC<mD$.

It is not immediately evident just exactly what these three conditions mean in practice, but a slight reformulation will make this clear. The conditions are, in fact, equivalent to the following propositions:

4. $A/B>m/n \Rightarrow C/D>m/n$
5. $A/B=m/n \Rightarrow C/D=m/n$
6. $A/B<m/n \Rightarrow C/D<m/n$,

where each of the implications is to be valid for all pairs, m, n, of positive whole numbers. To illustrate, we consider the proposition $\sqrt{3}:1::\sqrt{2}:1$. We observe that for $m/n = 3/2$, we have $\sqrt{3}/1>3/2$, but $\sqrt{2}/1<3/2$ (that is, $2\sqrt{3}>3\times1$, but $2\sqrt{2}<3\times1$). Thus, condition 4 (condition 1) is not satisfied and we do not have four terms in proportion. The example makes clear that A:B::C:D will be a proportion when, and only when, the component ratios are in accord with the whole fabric of rational *lógos*. That is because each of the three conditions compares these component ratios with each and every rational *lógos* – that is, each and every m/n, where m and n are positive whole numbers.

Now, in order for the theory to be effective, it must provide the mathematician with effective means for demonstrating theorems. We will illustrate both how it does so and how it differs from the earlier theory, by comparing the proofs of the same theorem according to the two different theories. The theorem in question is

> Theorem. $x:y::z:t \Rightarrow x:z::y:t$.
> (Proportionals are also alternately proportional.)

We first give a demonstration of this theorem that is in the spirit[18] of pre-Eudoxian, Pythagorean mathematics. In this proof, the variables x, y, z and t are supposed to be positive whole numbers.

[16] Euclid, Book V, Definition 5.
[17] Euclid, Book V, Definition 6.

First Demonstration. Since x, y, z and t are in proportion, a submultiple of y measures x in the same way that the corresponding submultiple of t measures z. That is, y is decomposed into n parts, each part being y/n, and x is equal to m of these parts. Similarly, t is decomposed into n parts, each part being t/n, and z is equal to m of these parts. All this follows from the (pre-Eudoxian) definition of proportion. Thus, we have two equations: $x=m(y/n)$ and $z=m(t/n)$. Dividing equals by equals, we obtain $x/z=m(y/n)/m(t/n)=y/t$, which completes the proof.

The following demonstration is in the spirit[19] of the new Eudoxian mathematics. In this proof, the variables x, y, z and t are supposed to be positive real numbers.

Second Demonstration. Let $x/y=z/t$. Let m be any positive whole number, then clearly $x/y=mx/my$. Similarly, let n be any positive whole number, so that $z/t=nz/nt$. Consequently,

$$mx/my=nz/nt. \qquad (*)$$

Now, if

1. $mx>nz$, (*) can be true only in case $my>nt$;
2. $mx=nz$, (*) can be true only in case $my=nt$;
3. $mx<nz$, (*) can be true only in case $my<nt$.

Thus, by the (Eudoxian) definition of proportion, x:z::y:t.

The second demonstration is clearly more complex. Since, in the case of incommensurables, the second term (or one of its submultiples) of a ratio does not measure the first and so we cannot use a simple equation to fix the first in terms of the second. Rather, we must compare the two terms with all rational *lógoi* and show that the three conditions of the Eudoxian definition are satisfied. Regardless of the complications, however, the theorem was proven according to the new methods. The same is true of all the most important theorems of the pre-Eudoxian theory of proportion. They all carry over into the more generalized setting.

We are now in a better position to appreciate Eudoxus' accomplishment in his theory of proportion. In a nutshell, he made incommensurables reasonable. That is, he developed mathematical techniques that enabled erstwhile irrationality to be comprehended intelligibly. Further, he did so in a way that must have been highly valued by the Pythagoreans because the new *lógos* is parasitic on the old *lógos* of positive whole numbers – and this in two distinct ways. First of all, as we have seen above, the theory made incommensurables intelligible when they were in accord with the entire fabric of the original *lógos* of positive whole numbers. That is, the new ratios have to be compared with all the old style ratios in order to make them intelligible. Hence, the old *lógos* became the

[18] It is in the spirit of pre-Eudoxian mathematics in that it uses the older Pythagorean conception of ratio and proportion. Nevertheless, we freely avail ourselves here of modern algebraical methods in order to eliminate irrelevant details. Euclid's treatment of this theorem is Proposition 13 of Book VII.

[19] Analogous remarks as those made in the previous note apply here. Euclid's treatment of this theorem is Proposition 16 of Book V.

measure of the new *lógos*. Second, the theorems of the old theory of proportion are carried over into the new theory by the use of more complicated proofs. For both of these reasons, then, the arithmetic of the natural numbers could continue to be seen as the model of true knowledge. The new theory partakes of the rational character of its older Pythagorean counterpart because it has the same mathematical structure, albeit in a more complicated and abstract setting.

Extreme and Mean Ratio

According to Roger Herz-Fischler (1998), the Golden Mean was probably discovered by the early Pythagoreans in their investigations of the regular pentagon or the pentagram (starred pentagon). We, in fact, know that the early Pythagoreans investigated these figures intensively and Kurt von Fritz (1945) has argued persuasively that they had discovered incommensurability in these investigations. So the Herz-Fischler suggestion fits nicely into the general historical pattern.

The name "Golden Mean" is a latter invention. The ancient Greeks referred to the construction as the Extreme and Mean Ratio because, when a line segment is cut in a certain proportion, one segment will be the geometric mean and the other will be the small extreme. The proportion that generates the cut is as follows: the whole segment is to the large part as the large part is to the small part. Thus, we have

$$\underline{a \quad\quad / \quad b \quad} \, ,$$

where $(a+b)/a = a/b$. This, in turn, gives us the equation $a^2\text{-}ab\text{-}b^2=0$. Following the lead of our investigations up to this point, we let the small extreme be a unit (that is, $b=1$). This gives us the equation $a^2\text{-}a\text{-}1=0$, whose roots are $(1\pm\sqrt{5})/2$. Since the ancient Greeks did not have negative numbers, we choose the positive root and, clearing fractions, obtain the following basic division into Mean and Extreme Ratio:

$$\underline{1+\sqrt{5} \quad\quad / \quad 2 \quad} \, .$$

Now, we must recall that the division of the segment given above only exhibits explicitly the mean and the small extreme of the defining proportion ($a+b$:a::a:b). When we write out this proportion explicitly, we obtain $3+\sqrt{5}$: $1+\sqrt{5}$ :: $1+\sqrt{5}$: 2. Thus, it becomes immediately obvious that $3+\sqrt{5}$: $1+\sqrt{5}$ are in the Extreme and Mean Ratio and that the process can be repeated *ad infinitum* thereby producing the following arithmetical flux[20]:

$$2,\ 1+\sqrt{5},\ 3+\sqrt{5},\ 4+2\sqrt{5},\ 7+3\sqrt{5},\ 11+5\sqrt{5},\ \ldots$$

[20] An arithmetical flux consists of one or more seeds and a rule that generates the other elements of the flux. As a first approximation, the reader may think of a flux as a sequence. For more details regarding the arithmetical flux and its role in Pythagorean mathematics, see Erickson and Fossa (2001) or Erickson and Fossa (to appear).

This flux has many interesting properties. We mention here only two of them. First, the coefficients of the √5's are given by the Fibonacci flux[21]: 1, 1, 2, 3, 5, 8, 11, ... Second, the flux itself is a kind of generalized Fibonacci flux, since each new element is the sum of the two preceding elements. By construction, of course, any two adjacent elements in the flux are in Extreme and Mean Ratio, that is, letting t_i be the ith term of the sequence, $t_{i+1}/t_i=(1+\sqrt{5})/2$.

The flux given in the preceding paragraph can be extended to the left by using the Fibonacci relationship: $t_{i+1}=t_{i-1}+t_i$; hence, $t_{i-1}=t_{i+1}-t_i$. This gives us

$$\ldots, 3-\sqrt{5}, -1+\sqrt{5}, 2, 1+\sqrt{5}, 3+\sqrt{5}, 4+2\sqrt{5}, 7+3\sqrt{5}, 11+5\sqrt{5}, \ldots$$

All of the numbers in this flux are positive and its only rational member is 2. Thus, we can think of two as the center of the flux.[22]

In light of Eudoxus' new theory of proportion, it now makes sense to ask about a Divided Line cut in Extreme and Mean Proportion. That is, we have a line segment, $\underline{x\,/\,y\,/\,y\,/\,z}$, cut so that x:y::y:z::Golden Ratio. The first problem is to decide what numbers to put on this Line, since the ratio of any two adjacent numbers in the flux given above is the same as that of any other two adjacent numbers. Our observation that 2 can be thought of as the center of the flux might lead us to put 2 in the spot of the geometric mean, giving raise to the following Divided Line:

$$\underline{-1+\sqrt{5}\,/\,2\,/\,2\,/\,1+\sqrt{5}}.$$

Although this seems to be an intuitively appealing solution to the problem, it must be remembered that 2 is the only rational member of the flux and, thus, should occupy the most noble position in the Divided Line. That position is that of the small extreme – the far right hand position by our convention. Doing so gives us what we will refer to as the Golden Line:

$$\underline{3+\sqrt{5}\,/\,1+\sqrt{5}\,/\,1+\sqrt{5}\,/\,2}\ .$$

The Golden Line belongs to a class of Divided Lines that is wider than those that we have previously encountered. Thus we may ask whether the analogue of the Divided Line Algorithm applies to this wider class of Divided Lines. The Following result shows that it does:

> **The General Algorithm of the Divided Line**. (i) The sum of the elements of a Divided Line is to the sum of the large extreme and the

[21] For more on the Fibonacci flux in Pythagorean arithmetic, see Erickson and Fossa (to appear).

[22] There are only two distinct solutions to the equation $a^2=a+1$. The solution $a=1-\sqrt{5}$, $b=2$ generates the sequence ..., $-4-2\sqrt{5}$, $3+\sqrt{5}$, $-1-\sqrt{5}$, 2, $1-\sqrt{5}$, $3+\sqrt{5}$, $4-2\sqrt{5}$, ... This sequence can be thought of as being centered on 2 and has $t_{i+1}/t_i=(1-\sqrt{5})/2$. Nonetheless, the terms in this sequence are alternately positive and negative. When we think of the numerical values as representing the lengths of line segments, this solution is extraneous since it does not satisfy the original word problem: "The whole is to the large part as the large part is to the small part."

> mean as this sum is to the large extreme. (ii) The sum of the elements of a Divided Line is to the sum of the small extreme and the mean as this sum is to the small extreme.

To show this, it will suffice to show that the middle terms of the Line generated is the geometric mean of the extremes. Thus, we are given that $\underline{x / y / y / z}$ is a Divided Line, so that $y^2=xz$. Part (i) of the General Algorithm produces the Line $\underline{x+2y+z / x+y / x+y / x}$. It is now easy to show that $x+y$ is the geometric mean of the extremes on this Line:

$$\begin{aligned}(x+2y+z)x &= x^2+2xy+xz \\ &= x^2+2xy+y^2 \\ &= (x+y)(x+y).\end{aligned}$$

Part (ii) of the General Algorithm produces the Line $\underline{x+2y+z / y+z / y+z / z}$. Using once again the fact that $y^2=xz$, we have

$$\begin{aligned}(x+2y+z)z &= xz+2yz+z^2 \\ &= y^2+2yz+z^2 \\ &= (y+z)(y+z).\end{aligned}$$

Thus, the General Algorithm is correct.

In order to see how the General Algorithm compares with the original Algorithm of the Divided Line, we will investigate two special cases: repeated application of part (i) of the algorithms and repeated applications of part two of the algorithms.[23] Recalling that Primitive Lines have the form $\underline{n^2 / nm / nm / m^2}$, where n and m are natural numbers, prime to each other, we will set out the n, m values beside the Line generated by them. We start by showing the first few lines of the case in which part (i) of the algorithms is iterated:

Primitive Lines $\underline{n^2 / nm / nm / m^2}$	n	m	General Lines $\underline{x / y / y / z}$
$\underline{1 / 1 / 1 / 1}$	1	1	$\underline{3+\sqrt{5} / 1+\sqrt{5} / 1+\sqrt{5} / 2}$
$\underline{4 / 2 / 2 / 1}$	2	1	$\underline{7+3\sqrt{5} / 4+2\sqrt{5} / 4+2\sqrt{5} / 3+\sqrt{5}}$
$\underline{9 / 6 / 6 / 4}$	3	2	$\underline{18+8\sqrt{5} / 11+5\sqrt{5} / 11+5\sqrt{5} / 7+3\sqrt{5}}$
$\underline{25 / 15 / 15 / 9}$	5	3	$\underline{47+21\sqrt{5} / 29+13\sqrt{5} / 29+13\sqrt{5} / 18+8\sqrt{5}}$
$\underline{64 / 40 / 40 / 25}$	8	5	$\underline{123+55\sqrt{5} / 76+34\sqrt{5} / 76+34\sqrt{5} / 47+21\sqrt{5}}$

In the case of the Primitive Lines given above, it is immediately evident that the generators, n and m, form Fibonacci sequences (the n values form a truncated Fibonacci sequence, in that the first 1 is missing). This sequence reappears in the General Lines, albeit in a zigzag fashion. Let m_i be the coefficient of the $\sqrt{5}$ in the expression for mean on the ith line and let f_i be the coefficient of the $\sqrt{5}$ in the expression for the large extreme on the ith line. Then, m_1, f_1, m_2, f_2, m_3, f_3, ... is the Fibonacci sequence. The sequence is also found in zigzag fashion in the coefficients of the small extreme (denoted by e_i) and the mean: (e_1), m_1, e_2, m_2, e_3, m_3, ... In this case, the sequence is augmented in that we have the extra term $e_1=0$.

Thus, there is a kind of similarity in the two fluxes of Lines shown above that could be interpreted as was Eudoxus' theory of proportion: the mathematical structure of

[23] For more details about some special cases of Primitive Lines, see Erickson and Fossa (to appear).

the flux of General Lines is parasitic on that of the Primitive Lines.[24] That said, however, we must also note an important disanalogy. In the case of the Primitive Lines, each new member of the flux is a new Primitive Line. In the case of these General Lines, in contrast, each new member of the flux is a multiple of the preceding member. In fact, each new member is $(3+\sqrt{5})/2$ times the preceding member. Thus, we have the following situation:

$$\underline{7+3\sqrt{5}\ /\ 4+2\sqrt{5}\ /\ 4+2\sqrt{5}\ /\ 3+\sqrt{5}} = (3+\sqrt{5})/2 \times \underline{3+\sqrt{5}\ /\ 1+\sqrt{5}\ /\ 1+\sqrt{5}\ /\ 2}$$

$$\underline{18+8\sqrt{5}\ /\ 11+5\sqrt{5}\ /\ 11+5\sqrt{5}\ /\ 7+3\sqrt{5}} = (3+\sqrt{5})/2 \times \underline{7+3\sqrt{5}\ /\ 4+2\sqrt{5}\ /\ 4+2\sqrt{5}\ /\ 3+\sqrt{5}}$$
$$= [(3+\sqrt{5})/2]^2 \times \underline{3+\sqrt{5}\ /\ 1+\sqrt{5}\ /\ 1+\sqrt{5}\ /\ 2}$$

$$\underline{47+21\sqrt{5}\ /\ 29+13\sqrt{5}\ /\ 29+13\sqrt{5}\ /\ 18+\ 8\sqrt{5}} = (3+\sqrt{5})/2 \times \underline{18+8\sqrt{5}\ /\ 11+5\sqrt{5}\ /\ 11+5\sqrt{5}\ /\ 7+3\sqrt{5}}$$
$$= [(3+\sqrt{5})/2]^3 \times \underline{3+\sqrt{5}\ /\ 1+\sqrt{5}\ /\ 1+\sqrt{5}\ /\ 2}$$

etc.

Whether this would have been seen as a new mathematical structure, interesting in its own right, or whether it would have been seen as defective in comparison with the structure of the Primitive Lines is open to question.

We now turn to the special case, in which part (ii) of the two algorithms is iterated. The first few lines of this case are:

Primitive Lines			General Lines
$\underline{n^2\ /\ nm\ /\ nm\ /\ m^2}$	n	m	$\underline{x\ /\ y\ /\ y\ /\ z}$
$\underline{1\ /\ 1\ /\ 1\ /\ 1}$	1	1	$\underline{3+\sqrt{5}\ /\ 1+\sqrt{5}\ /\ 1+\sqrt{5}\ /\ 2}$
$\underline{4\ /\ 2\ /\ 2\ /\ 1}$	2	1	$\underline{7+3\sqrt{5}\ /\ 3+\sqrt{5}\ /\ 3+\sqrt{5}\ /\ 2}$
$\underline{9\ /\ 3\ /\ 3\ /\ 1}$	3	1	$\underline{15+5\sqrt{5}\ /\ 5+\sqrt{5}\ /\ 5+\sqrt{5}\ /\ 2}$
$\underline{16\ /\ 4\ /\ 4\ /\ 1}$	4	1	$\underline{27+7\sqrt{5}\ /\ 7+\sqrt{5}\ /\ 7+\sqrt{5}\ /\ 2}$
$\underline{25\ /\ 5\ /\ 5\ /\ 1}$	5	1	$\underline{43+9\sqrt{5}\ /\ 9+\sqrt{5}\ /\ 9+\sqrt{5}\ /\ 2}$
$\underline{36\ /\ 6\ /\ 6\ /\ 1}$	6	1	$\underline{63+11\sqrt{5}\ /\ 11+\sqrt{5}\ /\ 11+\sqrt{5}\ /\ 2}$

In contrast to the previous case, this case generates new General Lines; consequently, only the first is the Golden Line. Both the Primitive Lines and the General Lines exhibit notable patterns. In fact, the patterns allow us to dispense with the rule of generation because we can continue to fill out the table by following these patterns. For the Primitive Lines, the pattern is especially simple. The generators, n, are just the sequence of natural numbers and m is constant, being always 1. Thus, the large extreme of the ith Line is just i^2, while its mean is i and its small extreme is 1. For the General Lines, the small extreme is the constant 2. The whole number part of the means form the sequence of odd numbers and the coefficient of the $\sqrt{5}$'s is always 1. The coefficients of the $\sqrt{5}$'s in the large extreme again form the sequence of the odd numbers. The pattern in the whole number part of the large extremes is not as easy to see, but, if we subtract 3 (the starting point) from each of these numbers, it becomes obvious that we are proceeding by multiples of 4. In fact, we are adding multiples of 4 in their natural order, thus we can use the triangular numbers (1, 3, 6,

[24] In fact, by looking at a more abstract flux, we could exhibit the reason why the Fibonacci sequence appears in both fluxes given above. We do not do so because it does not seem relevant to the argument.

10, 15, 21, ...) to express the result. Let w_i be the whole number part of the large extreme on the *i*th Line and let t_i be the *i*th triangular number. Then, $w_{i+1}=3+4t_i$.

This case shows, even more clearly than the previous case, that irrational magnitudes can form beautiful patterns that result in intelligible mathematical structures. It also shows that the intelligibility of this structure is parasitic on that of the natural numbers, since the basic patterns are here all set out in terms of the natural numbers (constants, odd numbers and triangular numbers).

Conclusion: The Analogous Line

Let us step back for a moment to review Plato's development as a thinker. This will help us to see how proportion became analogy and what this meant for Plato's philosophy.

The whole point of the early dialogues was to find universal definitions. The particular definitions sought after – virtue, for example – and much of the give and take of these dialogues reveals the strong influence of Socrates on the youthful Plato. Nevertheless, the actual procedure used for trying to find these definitions, eristic, was a common pastime in Classical Athens. Eristic itself, as a procedure, has a rather mathematical air about it. One of the contenders proffers a definition and his adversary tries to come up with a counterexample. If a counterexample is found, the first contender reformulates his definition to meet the objection. Then his adversary tries to come up with a new counterexample. The process continues until no new counterexample is found or one of the contenders tires of playing word games. There seems, however, to be a growing awareness in these early dialogues, that eristic, when taken seriously, can be a dialectical process that leads one to the truth.

The middle period of Plato's thought is characterized by the logical forms. These are hypostatizations of universal terms – mostly common nouns – that are supposed to account for our knowledge of what is really real. Plato eventually saw that there were apparently insuperable difficulties in this position and, thus, abandoned it.

In his third and final period, Plato accepted the Pythagorean thesis that the *lógos*,[25] that is the intelligibility, of the world is due to its mathematical structure. The role played by the logical forms in the middle period is now given to the mathematical forms and their instantiations in the material world as the cosmological forms. Then, following Eudoxus' new theory of proportion, he tries to think the *lógos* as generalized. Mathematical knowledge itself, however, is hypothetical ("if..., then...") and needs a point of departure. This is given by intuition, a mystical experience that cannot be had on demand, but that can be prepared for through dialectic thought.

Is there a connection between universal definition, logical forms of universals and generalized *lógos*, other than the very universality? It is clear that Plato is searching for true knowledge. The Parmenideian critique implied that we cannot have true knowledge of particulars and, therefore, the youthful Plato looked to universal terms. If these are to be

[25] *Lógos* is a meaning rich word. In ancient mathematical contexts it meant ratio and we have used it in that sense through most of this article. Here, however, we use it in its more philosophical sense of intelligibility.

other than empty words, however, they must refer to something. This led Plato on to his middle period, the one in which Plato, in Aristotle's phrase, was the "friend of the forms".

Finally, the search for true knowledge led Plato to the Pythagoreans. Mathematics seemed indeed to be the prototype of true knowledge. As Plato began to investigate the rich structural relations of such mathematical theories as that of the Divided Line, he must have become more and more convinced that mathematics is the key to rational intelligibility. But the old Pythagorean doctrine of reducing all being to relations between numbers was ultimately unsatisfying. Something more was needed.

Plato established the Academy and took on, as an associate, or perhaps co-founder, the mathematician Eudoxus. While at the Academy,[26] Eudoxus developed his new theory of proportion, extending *lógos* from whole numbers to the realm of erstwhile irrational magnitudes. This enabled Plato to develop, in two ways, a much more satisfying version of Pythagoreanism than had been done before.

First of all, Eudoxus' theory of proportion permitted the investigation of mathematical structures, like General Divided Lines, in which incommensurable quantities were admitted. The resulting structures were beautiful and intelligible and, thus, a truly possible part of the world. But, it also permitted Plato, in his relentless effort to think the universal, to use the segments of the Divided Line to represent, not only number and those strange incommensurable magnitudes, but also ontological and epistemological categories. Thus, 'ανάλογον (proportion) began to move closer to what we now call analogy. More importantly, however, it meant that these categories could be interpreted as having the mathematical structure inherent in the Divided Line itself. '*Análogon*, therefore, shows that mathematics is the *lógos* of the world, by revealing the mathematical structure hidden in the phenomena of the world. Through this understanding of analogy, Plato makes mathematics the model of intelligibility by imposing a mathematical model of discourse on non-mathematical contexts. We posit that the first step in this direction was the recognition, due to the structures developed from the Golden Line, that a non-*arithmòs lógos* can be beautiful.

The second way in which Eudoxus' new theory of proportion allowed Plato to develop a deeper and more satisfying Pythagoreanism was the way in which the new *lógos* was seen to be parasitic on the old *arithmòs lógos*. As we have seen, this was true not only for Eudoxus' own theory, but also for the interesting structures forthcoming from the Golden Line. This allowed Plato to see the world as consisting of a basic reality from which other realities emerged in dependently hierarchical structures. The Divided Line itself depicts this hierarchical structure, but it is doubtlessly only the overall structure, from which most of the details have been omitted.[27]

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[26] See, for example, Lasserre (1966).
[27] For a discussion of how each segment of the Divided Line is a Divided Line in its own right, see Erickson and Fossa (to appear).

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