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CHARLES PEIRCE AND BERTRAND RUSSELL ON EUCLID

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[CHARLES PEIRCE E BERTRAND RUSSELL SOBRE EUCLIDES]

§0. Introduction

Both Charles Sanders Peirce (1839–1914) and Bertrand Russell (1872–1970) held that Euclid's proofs in geometry were fundamentally flawed, and based largely on mathematical intuition rather than on sound deductive reasoning. They differed, however, as to the role which diagramming played in Euclid's demonstrations. Specifically, whereas Russell attributed the failures on Euclid's proofs to his reasoning from diagrams, Peirce held that diagrammatic reasoning could be rendered as logically rigorous and formal. In 1906, in his manuscript "Phaneroscopy" of 1906, he described his existential graphs, his highly iconic, graphical system of logic, as a moving picture of thought, "rendering literally visible before one's very eyes the operation of thinking *in actu*", and as a "generalized diagram of the Mind" (Peirce *1906*; *1933*, 4.582). More generally, Peirce personally found it more natural for him to reason diagrammatically, rather than algebraically. Rather, his concern with Euclid's demonstrations was with its absence of explicit explanations, based upon the laws of logic, of how to proceed from one line of the "proof" to the next. This is the aspect of his criticism of Euclid that he shared with Russell; that Euclid's demonstrations drew from mathematical intuition, rather than from strict formal deduction.

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[1] The esteemed scholar and historian of logic Dr. Irving H. Anellis passed away on the 15th of July 2013. We are proud to publish as posthumous this paper that is one of the last works of this outstanding scholar. Paper submitted as an invited plenary lecture given during the 10th National Seminar on the History of Mathematics at UNICAMP in Campinas, SP, Brazil, on March, 2013.

§1. The Problem

For some two thousand years, until well into the 19th century, Euclid's *Elements* were putatively conceived as providing the most perfect example of rigorous deductive reasoning. In Peirce's (*1892*) words: "In the pre-Lobachevskian days, elementary geometry was universally regarded as the very exemplar of conclusive reasoning carried to great lengths. It had been the ideal of speculative thinkers in all ages." Plato is supposed, for example, to have inscribed over the gate of his Academy: "Let no one ignorant of geometry enter" — *ΑΓΕΩΜΤΡΗΤΟΣ ΜΝΔΕΙΣ ΕΙΣΙΤΩ*. There was, however, a recognition, almost from the outset, that there were difficulties involving Euclid's fifth, parallel postulate.

Euclid himself apparently recognized that his fifth postulate was somehow of a different character than the remainder of the postulates provided in his axiom system, since he avoided its use except in extremity. It remained untouched through Euclid's proofs of his first twenty propositions. From Euclid onward, mathematicians sought to provide a proof, within Euclid's axiomatic system, of the parallel postulate. In contemporary terms, they sought to demonstrate that the parallel postulate is independent in Euclid's axiomatic system. No one, however, considered the possibility that the parallel postulate is *inconsistent*, and Immanuel Kant (1724–1804), for example, held that the only *possible* geometry is a geometry in which the parallel postulate holds, that is, that the only possible geometry is Euclidean geometry, and that space is Euclidean.[2]

Thus, efforts were undertaken to make the minor repairs that would rescue Euclid from this one special and particular difficulty. Thus, for example, we have Proclus Diadachos (411–485 C.E.), in his commentary on Euclid, who wrote (see Proclus *1970*, pp. 150–151) that:

> *"This [fifth postulate] ought even to be struck out of the Postulates altogether; for it is a theorem involving many difficulties which Ptolemy, in a certain book, set himself to solve, and it requires for the demonstration of it a number of definitions as well as theorems. And the converse of it is actually proved by Euclid himself as a theorem. It may be that some would be deceived and would think it proper to place even the assumption in question among the postulates as affording, in the lessening of the two right angles, ground for such an instantaneous belief that the straight lines converge and meet. To such as these Geminus correctly replied that we have learned from the very pioneers of this science not to have any regard to mere plausible imaginings when it is a question of the reasonings to be included in our geometrical doctrine. For Aristotle says that it is as justifiable to ask scientific proofs of a rhetorician as to accept mere plausibilities from a geome-ter; and Simmias is made by Plato to say that he recognizes as quacks those who*

[2] See. e.g. (Anellis *1991*).

> *fashion for themselves proofs from probabilities. So in this case the fact that, when the right angles are less-ened, the straight lines converge is true and necessary; but the statement that, since they converge more and more as they are produced, they will sometime meet is plausible but not necessary, in the absence of some argument showing that this is true in the case of straight lines. For the fact that some lines exist which approach indefinitely, but yet remain non-secant, although it seems improbable and paradoxical, is nevertheless true and fully ascertained with regard to other species of lines [for example curves like the hyperbola that has asymptotes]. May not then the same thing be possible in the case of straight lines that happens in the case of the lines referred to? In-deed, until the statement in the Postulate is clinched by proof, the facts shown in the case of other lines may direct our imagination the opposite way. And, though the controversial arguments against the meeting of the straight lines should contain much that is surprising, is there not all the more reason why we should expel from our body of doctrine this merely plausible and unrea-soned (hypothesis)?*
> *It is then clear from this that we must seek a proof of the present theorem, and that it is al-ien to the special character of Postulates. But how it should be proved, and by what sort of argu-ments the objections taken to it should be removed, we must explain at the point where the writer of the Elements is actually about to recall it and use it as obvious. It will be necessary at that stage to show that its obvious character does not appear independently of proof, but is turned by proof into matter of knowledge."*

There is the 13th-century example occurs in the *Perspectiva* (*ca*. 1274) of Witelo (also Erazm Ciolek Witelo; Witelon; Vitellio; Vitello; Vitello Thuringopolonis; Vitulon; Erazm Ciołek; *ca*. 1230–*ca*. 1280)[3]; and, most famously Giovanni Saccheri's (1667–1733) employment in his (*1733*) *In Euclides ab Omni Nævo Vindicatus* to rescue the fifth postulate in of the *consequentia mirabilis* that he claimed to have rediscovered and presented in his popular textbook (*1692*) *Logica demonstrativa*. Saccheri's *Logica* had the intentions of applying what he considered were the strict standards of geometrical proofs to logic and of reducing the number of "first principles" to a minimum. His logic is the syllogistic of Aristotle, and his approach is deductive, following the methodology of Euclid's *Elements*. Saccheri used the *consequentia mirabilis* to "prove" that, if Euclid's parallel postulate is false, then would yield a contradiction. He wrote of the *consequentia mirabilis* (*1733*; see Saccheri *1920*, p. xxii):[4]

[3] See (Unguru *1978*) and (Witelo *1991*).
[4] See (Euclid *1926*, vol. 2, pp. 397–398) for Euclid's proof, with T. L. Heath's exposition at (Euclid *1926*, vol. 2, pp. 398–399). For discussions, see also, e.g. (Angelelli *1975*; *1995*), (Dou *1970*), and (Hoormann *1976*).

> *"Now in fact we may conceive another way of proceeding, and as I think a beautiful way, by which I demonstrate the same truths without the assistance of the postulate. I shall proceed thus: I shall assume the contradictory of the proposition to be demonstrated and from it I shall elicit directly the proposition to be proved. This method of proof was used by Euclid in Book IX, Proposition 12".*

Using the *Saccheri quadrilateral*,

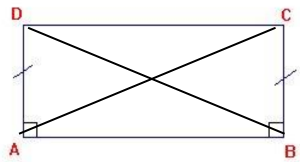

Figure 1.

Saccheri examined three possible cases: the Hypothesis of the Right Angle (where angle C is a right angle and the summit angles are = 90°), the Hypothesis of the Acute Angle (where angle C is an acute angle and the summit angles are < 90°), and the Hypothesis of the Obtuse Angle (where angle C is an obtuse angle and the summit angles are > 90°), he attempted to disprove both the Hypothesis of the Acute Angle and the Hypothesis of the Obtuse Angle, and by elimination thereby prove the Euclid's Parallel Postulate, since the three are incompatible. His first step was to prove that indeed the three hypotheses are mutually exclusive. He easily disproved the Hypothesis of the Obtuse Angle, since it violates Euclid's postulate that a straight line is of infinite length. He next attempted to also disprove the Hypothesis of the Acute Angle. Along the way, he proved, among other characteristics of the three resulting geometries, that under the Hypothesis of the Acute Angle the sum of the angles of a triangle are less than 180 degrees; under the Hypothesis of the Right Angle, equal to 180 degrees, and, Hypothesis of the Obtuse Angle, greater than 180 degrees. It was left to later geometers to show that Saccheri had failed, since subsequent analysis of his arguments revealed that, with respect to his treatment of Hypothesis of the Obtuse Angle, his reasoning was sloppy and in parts incorrect. (Saccheri's argument runs: $\Delta ABC \cong \Delta BAC$ (where both sides and the angle are included). Therefore $AC = BD \Rightarrow \Delta ADC \cong \Delta BCD$ / $\therefore \angle ADC = \angle BCD$.)

As (Coolidge *1963*, p. 69) said of Saccheri, "this careful logician undertook to prove the correctness of Euclid's postulate by showing that when it is replaced by another, a contradiction is sure to arise." By showing that the
first four Euclidean postulates together with the *negation* of the fifth postulate yields a contradiction, Saccheri would have proven that the first four postulates, together with the fifth postulate, is a valid system. This is precisely what Saccheri thought he did — *vindicated* Euclid by proving that the assumption of the negation of the fifth postulate

together with the first four postulates yields a contradiction. In particular, he considered an isosceles birectangle *ABDC*, a quadrilateral in which $AC = BD$ and angles A and *B* are right angles. Using only the first four postulates and the first twenty-eight theorems of Euclid which may be derived from them without the aid of the fifth postulate, Saccheri easily showed that angles *C* and *D* are equal to each other. Using an assumption invalid under the obtuse angle hypothesis (that straight lines are infinitely long), but without the aid of the fifth postulate, Saccheri was able to eliminate the possibility that angles *C* and *D* are obtuse. But he was unable to eliminate the possibility that they might be acute. To do so would have in fact required the fifth postulate. The best that Saccheri could do was employ some questionable characterizations of infinity to derive an unconvincing and irrelevant contradiction to eliminate the possibility that the angles are acute. (Eves *1981*, p. 69) has gone so far as to express his opinion that Saccheri himself was not fully convinced by his argument. (Dou *1970*) gives a detailed analysis of Saccheri's arguments and discusses Saccheri's possible influence on the subsequent development of non-Euclidean geometry.)

Most of those of Saccheri's and Kant's contemporaries working on the parallel problem, like Saccheri himself, in fact really thought that they had shown that the fifth postulate does follow from the remainder of Euclid's axioms, and all of them were attempting, like Saccheri, to find such a proof.

These problems, and in particular efforts to demonstrate Euclid's parallel postulate, led ultimately to the discovery and recognition, if not immediately and unconditionally, of non-Euclidean geometries, sealed by the relative consistency proof of Eugenio Beltrami (1835–1900) in his (*1868*) "Saggio di interpretazione della geometria non-euclidea", demonstrating that, Beltrami demonstrated that non-Euclidean geometry (in the particular instance hyperbolic geometry) is consistent if non-Euclidean geometry is consistent.[5] Beltrami, that is, was able to use Saccheri's proof of the independence of the parallel postulate to develop the concept of *relative consistency proof* for non-Euclidean geometry, showing that non-Euclidean geometries are inconsistent only if Euclidean geometry is inconsistent, or conversely, that if Euclidean geometry is consistent, then so are non-Euclidean geometries. But to impute to Saccheri himself — or to Kant — the view that Saccheri proved that non-Euclidean geometries are possible,[6] is anachronistic, and thus a misunderstanding of the history of geometry. Under these circumstances, it is difficult to believe that Russell could have concluded that Kant could have believed in the possibility of non-Euclidean geometries. Kant came as close to the subject of non-Euclidean

[5] (Bonola *1955*) remains the most accessible source for the chief publications of Bolyai and Lobachevskii in non-Euclidean geometry in English translation, and (Sommerville *1911*) the standard bibliography of works in non-Euclidean geometry up to its publication.

[6] As (Watling *1990*) did. This argument is based upon the debatable supposition (compare (Martin) and (Fang) that Kant was not ony aware of Saccheri's work, indeed corresponded with him, but moreover held with Saccheri proved exactly the opposite of what he in fact thought and claimed to have proven, namely that denial of Euclid's parallel postulate leads to a contradiction.

Against this historical background, let us return to consideration of Peirce's and Russell's views of Euclid and his work. We can attend to their view in part with a consideration of their responses to Kant's philosophy of mathematics and philosophy of logic, which may be briefly summarized, with respect to geometry by the proposition that the propositions of geometry to be *à priori* synthetic, and that time and space are absolutes, conditions of the understanding for the possibility of perception.

§2. Russell on Euclid.

In *An Essay on the Foundations of Geometry*, Russell (*1897*, pp. 54–63) argued that the development of "metageometry", i.e. the axiomatic foundations of (Euclidean and non-Euclidean) geometries, has shown that Kant's argument for the apodeiciticity of Euclidean geometry breaks down. But Russell did not accept either the position that non-Euclidean geometries are necessary (in any Kantian sense). Instead, Russell (*1897*, p. 6) concluded that only those axioms which are common to both Euclidean and non-Euclidean geometries are *à priori*, whereas the axioms specific to Euclidean geometry are "wholly empirical", as are those axioms specific to the various non-Euclidean geometries. Moreover, the claim (made by John Watling (1990) that Kant was unaware of the possibility of non-Euclidean geometries is unfair to the editor also with respect to specific arguments made by Russell. In *An Essay on the Foundations of Geometry* (*1897*, pp. 54-63), Russell argued that the development of "metageometry", i.e. the axiomatic foundations of (Euclidean and non-Euclidean) geometries, has shown that Kant's argument for the apodeiciticity of Euclidean geometry breaks down. But Russell did not accept either the position that non-Euclidean geometries are necessary (in any Kantian sense). Instead, Russell (*1897*, p. 6) concluded, *contrary* to Watling's (and Gottfried Martin's, *et aliia)* interpretation of Kant's position, that only those axioms which are common to both Euclidean and non-Euclidean geometries are apriori, whereas the axioms specific to Euclidean geometry are "wholly empirical," as are those axioms specific to the various non-Euclidean geometries.

This view was reinforced by Russell's (*1898*) reply to Louis Couturat's (1868–1914) (*1898*) review of the *Essay,* in which Russell first accepts Couturat's assertion that Russell's argument in the *Essay* for the empirical character of Euclidean geometry is weak but then defends the empiricality of Euclidean geometry with new arguments. Russell (*1902*, p. 673) made the point much more clearly in his *Encyclopedia Britannica* article on non-euclidean geometry that, although Kant's view that geometry is synthetic permits the possibility that there might be non-euclidean geometries, saying that "Kant maintained [that there] is *à priori* ground for excluding all or some of the non-Euclidean spaces." Whether Kant knew of Saccheri's work or knew Saccheri remains an open question. In any case, Kant did not entertain the possibility of non-Euclidean geometries. Carl Friedrich Gauss (1777–1855), who worked on the problem of the parallel postulate and became convinced that alternatives were possible, but decided not to publish for fear of his reputation, was able to convince János Bolyai (1802–1860) to publish his (*1832*) result, which subsequently became the first instance of a published presentation of a non-Euclidean geometry. (Russell's (*1897*, p. 7) assertion that Adrien Marie Legendre (1752–1833) was the first to refuse to accept the parallel postulate without a proof, and the first to attempt to deduce it from the others, is false.[7]) Bolyai wrote to his father in 1823 that "I have discovered things so wonderful that I was astounded ... out of nothing I have created a strange new world." It took two more years before he had it completely written out. His essay on non-Euclidean geometry eventually appeared as a twenty-four-page appendix to his father's book, *Tentamen juventutem studiosam in elementa matheseos purae*. Gauss, corresponded with Kant, and Gauss went so far as to privately express the view that, when it came to his philosophy of mathematics, Kant was hopelessly muddled. He wrote (*1860-65*, vol. IV, p. 337), in his now infamous letter of November 1, 1844 to the astronomer Heinrich Christian Schumacher (1780–1850), that "you see the same sort of [mathematical incompetence] in the contemporary philosophers.... Don't they make your hair stand on end with their definitions? ...Even with Kant himself it is often not much better; in my opinion his distinction between analytic and synthetic propositions is one of those things that either run out in a triviality or are false."

Nikolai Ivanovich Lobachevskii's (1792–1856) non-Euclidean geometry followed. The "imaginary" geometry or "pan-geometry" of Lobachevskii first appeared in print in his multi-part "Elements of Geometry" (*1829-30*); his "Géometrie imaginaire" (*1837*) and its Russian original (*1835*) is a sustained and detailed account, and his *Geometrische Untersuchungen zur Theorie der Parallellinien*. (*1840*) and (*1855*) *Pangéométrie*, his best known works, being summaries. All straight lines which in a plane go out from a point can, with reference to a given straight line in the same plane, be divided into two classes, *viz.*, into cutting and non-cutting. The boundary lines of the one and the other class of those lines will be called parallel to the given line. Thus Lobachevskii replaced Euclid's fifth postulate with the postulate that asserts that *there exist two lines parallel to a given line through a given point not on the line*. He went on to develop various trigonometric identities for triangles which held in this geometry, showing that as the triangle became small the identities tended to the usual trigonometric identities. The pangeometry is a general system in which Euclid's fifth postulate does not necessarily hold, and Euclidean geometry is treated as a special case.

By the time Peirce and Russell were working, mathematicians, if not all philosophers, seriously considered the possibility that Euclid's fifth postulate need not necessarily hold, and that alternative, non-Euclidean geometries, could not only be constructed, but could had to be given as both empirical and formal status.[8] One of the aims of Russell's *Essay on the Foundations of Geometry* was to explore, within a neo-Kantian framework, meta-geometry as the set of most basic axioms necessary and sufficient for the

[7] Adrien-Marie Legendre (1752–1833) spent some twenty years working on the problem of parallels and his results are scattered through the several editions (from 1794 to 1823) of his (*1794*) *Éléments de Geometriè.* These results were finally brought together in Legendre's (1833) *Reflexions sur différentes manières de démontrer la théorie des parallèles ou le théorème sur la somme des trois angles du triangle,* Mémoires de l'Académie royale des Sciences de l'Institute de France XΠ (1833), to which Russell referred.

[8] See (Gray *2007*) for a history of geometry in the 19th century and (Rosenfeld *1988*) for a general history of non-Euclidean geometry.

possibility of all geometries, Euclidean and non-Euclidean. In *pangeometry* or *metageometry*, that is, in the abstract geometry that admits the possibility of both Euclidean geometry and non-Euclidean geometries, we can classify three types of space: Given a line *l* and a point *P* not on the line, the elliptic parallel property contrasts with the Euclidean and hyperbolic parallel properties:

> *Elliptic:* any line through *P* meets *l* in just one point (Bolyai).
> *Euclidean (parabolic)*: just one line through *P* may be found, which does not meet *l* (Euclid).
> *Hyperbolic*: more than one line through *P* may be found, which do not meet *l* (Lobachevskii).

Neither Bolyai nor Lobachevskii were able to arrive at a proof of the consistency of their respective non-Euclidean geometries, however, any more than mathematicians since Euclid could prove the consistency of Euclid's. This is what led Beltrami to devise his relative consistency proof of 1868, demonstrating that, if Euclid's system was consistent, then so was Lobachevskii's. Meanwhile, Bernhard Riemann (1826–1866) in his *Habilitationsvortag* "Über die Hypotheses, welche der Geometrie zu Grunde liegen" (read in 1854, published in 1868), reformulated the whole concept of geometry which he saw as a space with enough extra structure to be able to measure things like length. One of his tasks in this lecture was to deal with the issue of how to define an *n*-dimensional space, what today we call a Riemannian space.

§3. Peirce on Euclid

Turning specifically to the details of their respective criticisms of Euclid, we find Peirce, in his (*1892*) review, "The Non-Euclidean Geometry", of the English translation (Lobachevskii *1891*; *1892*) by George Bruce Halsted (1853–1922) as of the second (*1887*) edition Lobachevski's *Geometrische Untersuchungen zur Theorie der Parallellinien*, Peirce described Euclid's *Elements* as "unmathematical", writing in his review that: "The truth is that elementary geometry, instead of being the reflection of human reasoning, is riddled with fallacies, and is thoroughly unmathematical in its development. It has in the same measure confused all mathematics, by leaving unnoticed most of the really fundamental propositions, while raising to an undue rank certain others almost arbitrarily selected…." In the manuscript "The Non-Euclidean Geometry Made Easy", dating from the late Spring of 1890 (*1890*; see Peirce *2010a*, pp. 25–29) considered that Euclidean intuition may be misleading from the logical perspective. He notes (*2010a*, p. 25) that: "We have an *a priori* or natural idea of space, which by some kind of evolution has come to be very closely in accord with observations," and immediately offers a caveat: "But we find in regard to our natural ideas, in general, that while they do accord in some measure with fact, that by no means do so to such a point that we can dispense with correcting them by comparison with observations." Thus, Peirce would seem to be in agreement with that part of Russell's position in the *Foundations of Geometry* (*1897*, p. 6) according to which Euclidean geometry is empirical, Peirce offering the following case (*2010a*, p. 25), working with the diagram:

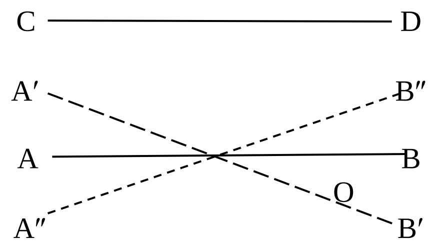

Figure 2.

> Given a line CD and a point O. Our natural (Euclidean) notion is that
> 1st there is a line AB through O in the plane OCD which will not meet CD at any finite distance from O.
> 2nd that if any line A′B′ of A″B″ through O in the plane OCD be inclined by any finite angle, however small, to AB, it will meet CD at some finite distance from O.

and asking (*2010a*, p. 25) whether this natural notion is "*exactly* true". His multi-level reply (*2010a*, pp. 25–26) is largely empirical, based upon epistemological, evolutionary, and statistical concerns, among the six listed being the admission: "It may be true, perhaps. But since the chance of this is as 1:∞ or $^0/_1$, the logical presumption is, and must ever remain, that it is not true," and contending that the first proposition, that it is simply uncertain, is sufficient to admit the possiblity of non-Euclidean geometry, explaining (*2010a*, p. 26) depending upon whether one denies either the 1st or 2nd proposition, one of two non-Euclidean geometries, either Lobachevskii's hyperbolic geometry or Bolyai's elliptical geometry. He illustrates by way of projective geometry, viewing the plane from an angle and noting that the parabolically parallel lines will either intersect at the origin only, or at some point at infinity. It is not surprising, therefore, that Peirce should have also made efforts at treating geometry metrically by introducing the absolute synthetically (see, e.g. *1883?*, the manuscript "Non-Euclidean Geometry", of *circa* 1883 or later). Indeed, in "The Logic of Continuity" (see Peirce *2010b*, p. 181), he calls elementary geometry "nothing but the introduction to *geometrical metric*, or the mathematical part of the physics of rigid bodies."

Peirce essential point is that, although the true geometry, whether Euclidean or non-Euclidean, is the actual geometry of space, is an empirical question, the construction of any of these possible geometries is possible from application of sound logical procedures and mathematical reasoning that has already been developed, such as projective geometry. In the manuscript "The Axioms of Geometry" of Spring 1873 (Peirce *1986*, pp. 189–190), for example, Peirce offers five axioms for a general three-dimensional geometry:

1. All points of space have the same properties.
2. All pairs of points have the same properties.
3. Space has three dimensions.
4. Space permits parallel motion.
5. Space permits rotation.

Clearly, axiom 4, in permitting, but not requiring, parallel motion, is tantamount to permitting Euclidean and non-Euclidean geometries. Peirce adds, in discussing his fourth axiom, that geometry provides a means for determining whether motion is parallel. His corollary to the second axiom, that the sum of the three angles of a triangle is equal to two right angles adds the condition that this is the case because unless different lengths have different properties, a condition that would appear to open at least the possibility of the existence of elliptical and/or hyperbolic angles. Peirce's fifth axiom is the counterpart to Russell's (*1897*, pp. 149–160) Axiom of Free Mobility for metrical geometry, which allows pointwise transformations without loss of congruity.

For Charles Peirce, following on his father, Benjamin Peirce's (1809–1880) (*1870*) definition of mathematics as the science which draws necessary conclusion, mathematics consequently deals with hypotheses, and these hypotheses are, for Charles Peirce (*ca. 1895*), mental creations.

What emerges in Peirce's conception of mathematical reasoning. Cornelis De Waal (*2013*, p. 28) has noted that mathematical reasoning for Charles Peirce, presumably having geometry specifically in mind, "includes much more than giving proofs. Reducing mathematics to demonstration as Peirce sees Euclid do in the *Elements*, leaves out the most important aspects of mathematical reasoning: construction, observation, experiment, abstraction, and generalization." In short, for Peirce, according to De Waal, *proof* is what logically validates our geometrical intuition and experience. De Waal (*2013,* p. 28) goes on to quote from a letter of Charles Peirce of 18 November 1894 to his brother, Harvard University mathematician James Mills Peirce (1834–1906) stating that demonstration, prrof, is just "the pavement over which the mathematician drives his team with the goal in view and with a plan for reaching it." But, by the nature of the role of these other aspects of mathematical reasoning, some care must be taken by Peirce of the ontological status of geometric entities under investigation in a manner that is absent from Russell's (*1901*, pp. 83–84), where diagrams play only an ancillary role, as merely an *aide de mémoire*, and the proof have an altogether hypothetical nature, such that, while "[fo]rmerly, it was held by philosophers and mathematicians alike that the proofs in Geometry depended upon the figures" (Russell *1901*, 99); but

> *"nowadays, this is known to be false. In the best books there are no figures at all. The reasoning proceeds from a set of axioms laid down to begin with. If a figure is used, all sorts of things seem obviously to follow, which no formal reasoning can prove from explicit axioms, and which, as*

> *a matter of fact, are accepted only because they are obvious. By banishing the figure, it became possible to discover all the axioms that are needed; and in this way all sorts of possibilities, which would have otherwise remained undetected, are brought to light."*

With this in view, Peirce's attempts at a "New Elements of Mathematics (*Καινὰ Στοχεία*)"[9] was intended as a replacement, built upon the principles of formal logic, of Euclid's *Elements*, incorporating the development of non-Euclidean geometries.

§4. Conclusion

Turning back to Russell, then. we note his little-known paper "The Teaching of Euclid" (Russell *1902*), in which Russell took Euclid seriously to task for the lack of the "logical excellence" which Euclid was reputed to have presented in his *Elements*, and which had traditionally been ascribed to him since antiquity. Against the concept of Euclid's *Elements* as a masterpiece and exemplar of logical reasoning, because Euclid's "logical excellence is transcendent," Russell began in his essay "The Teaching Euclid" (*1902*, p. 165) by asserting that this claim "vanishes on a close inspection. His definitions to not always define, his axioms are not always indemonstrable, his demonstrations require many axioms of which he is quite unconscious. A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclid's earlier proofs fail before this text." Among the examples of problems are the first proposition, which assumes, without warrant, the intersection of the circles used in the construction; another example is the fourth proposition, which Russell calls "a tissue of nonsense", given that superposition is "a logically worthless device," and a logical contradiction arises when, taking the triangles as spatial rather than material, one engages the idea of moving them, while, if taking them as material, they cannot be supposed to be perfectly rigid and thus, when superposed, they are certain to be slightly deformed from their previous shape. Russell argued that some of Euclid's proofs were erroneous, and that some alleged demonstrations were not really even proofs at all, properly so-called. This is quite apart even from the question of the correctness and independence of Euclid's controversial Fifth Postulate, the Parallel Postulate, and the possibility or impossibility of non-Euclidean geometries. Rather, Russell's criticism of Euclid hinged upon the nature of proof.

By way of example, Russell explains, in *The Principles of Mathematics* (*1903*, §15, p. 14) that:

> *"The fifth postulate of Euclid follows from the fourth: if the fourth is true, so is the fifth, while if the fifth is false, so is the fourth. This is a case of material implication, for both propositions are absolute constants, not dependent for their meaning, not depending upon the assigning of a value*

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[9] Included in (Peirce *1975*, vol. 2).

> *to a variable. But each of them states a formal implication. The fourth states that if x and y are triangles fulfilling certain conditions, then x and y are triangles fulfilling certain other conditions, and that this implication holds for all values of x and y; and the fifth states that if x is an isosceles triangle, x has angles at the base equal."* [10]

In writing of Euclid in "The Teaching of Euclid" (*1902*), Russell pointedly insisted that Euclid's so-called logical proofs depended upon one's mathematical intuition, rather than rigorous formal deduction, and that the intuitive inferences drawn in Euclid relied almost entirely, if indeed not wholly, upon the construction of the diagrams. The search for the kind of absolute certainty, *in more geometrico* and the encounter with the failure of Euclid to provide logically valid proofs for his propositions, the reliance upon intuition and geometric construction rather than strictly deductive reasoning, as well as the more questions regarding Euclid's parallel postulate and the legitimacy of Euclidean and non-Euclidean geometries, is what drove Russell to logicism.[11]

It was, likewise, Peirce's interaction with non-Euclidean geometries, along with the Euclidean geometry as its rival, that drove his conclusion that Euclid is "riddled with fallacies…and is thoroughly unmathematical…." The major significant difference between Peirce and Russell is that, although both identified the problem as lack of logical rigor and the role of intuition, Russell, unlike Peirce, found the cause in the excessive reliance by Euclid and his successors upon geometrical constructions and the visual aspects of the diagrams employed.

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[10] The standard English translation, in the classic edition by Thomas Little Heath (1861–1940) for Euclid's fourth and fifth postulates are, respectively (Euclid *1926*, vol. 1, p. 154): "That all right triangles are equal to one another" and (Euclid *1926*, vol. 1, p. 155): That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles."

[11] See (Anellis *1995*) for the role which geometry played in Russell's rejection of idealism and adoption of logicism.

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